Baire-Class 1 and Continuity

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Notes for a 4/26/19 talk in the Portland State University Analysis Seminar on the continuity of derivatives, and more generally functions of Baire-Class 1, that followed up on Pieter VandenBerge's 4/12/19 talk [**VB**] on the possible discontinuities of derivatives. Pieter showed that every derivative has the "Intermediate Value Property;" Here we'll see that every derivative, and more generally, every function of Baire Class 1, is continuous on a dense G_δ set.

1 *Derivatives and Baire-Class 1*

Our setting will be a closed interval I of the real line $\mathbb{R}$. This interval could, for example, be the whole line, some half-line, e.g. $[0, \infty)$, or a finite interval, e.g., $[0, 1]$.[1]

[1] Except for our discussion of derivatives, I could even be a complete metric space.

A derivative on I is a function $f : I \to \mathbb{R}$ for which there exists a differentiable $F : I \to \mathbb{R}$ with $F' = f$ on I.

The Intermediate-Value Property (IVP) We say a function $I \to \mathbb{R}$ "has the IVP" if any real number between two values of f is also a value of f.

Darboux's Theorem.[**Da** (1875)] *Every derivative on I has the IVP.*

The IVP need not have anything to do with continuity; there are functions having the IVP that take every real value in every interval. A particularly interesting example is "Conway's base 13 function," see e.g., [**Om**] for this, and for an interesting discussion of the IVP

We learn early on in Calculus that every *continuous* function $I \to \mathbb{R}$ has the IVP. Might this signal that derivatives possess some vestige of continuity? We'll address this question next.

Baire-Class 1 is the collection of functions that are *pointwise limits of continuous functions*, i.e., all functions $f : I \to \mathbb{R}$ for which there exists a sequence (f_n) of functions continuous on I with

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \forall\, x \in I.$$

Proposition 1. *Every derivative on I is of Baire-Class 1.*

Proof. For $x \in I$:

$$f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{n\to\infty} \underbrace{n\Big[f\Big(x + \frac{1}{n}\Big) - f(x)\Big]}_{f_n(x)}$$

where each f_n is continuous on I. □

In these notes we'll prove

Baire's Continuity Theorem[2] *Every function of Baire-Class 1 on I (in particular: every derivative) is continuous on a dense subset of I.*

[2] [**Ba** (1899)]. This is not standard terminology.

Corollary 2. *Dirichlet's function ($\equiv 1$ on the rationals and 0 on the irrationals) is not of Baire-Class 1 on any interval.*

Proof. Dirichlet's function has no points of continuity. □

2 *Uniform Approximation*

UNLESS OTHERWISE NOTED, $f\colon I \to \mathbb{R}$ is of Baire-class 1, i.e., there exists a sequence (f_n) of functions continuous on I such that

$$f(x) = \lim_n f_n(x) \qquad \forall\, x \in I. \tag{1}$$

If the above convergence were *uniform* on I, then f would inherit the continuity of the f_n's at *every point* of I.

Now uniform convergence of f_n to f on I means that for each $\varepsilon > 0$ the continuous function f_n provides, for each n sufficiently large, an ε-approximation to f that is *uniform over* I. Does mere *pointwise* convergence bestow upon (f_n) some kind of uniform approximation?

To study this question, let's define, for *any* sequence of functions $(f_n)\colon I \to \mathbb{R}$, and any $\varepsilon > 0$, its set of "ε-Cauchy-ness"[3]

$$C(\varepsilon) = \bigcup_{N>0} \underbrace{\bigcap_{n,m\geq n} \left\{|f_n - f_m| \leq \varepsilon\right\}}_{=:F_N(\varepsilon)}. \tag{2}$$

[3] Once again: not standard terminology. However the abbreviation $\{g \leq \varepsilon\}$ *is* standard, and stands for the set $\{x \in I\colon g(x) \leq \varepsilon\}$, with similar notation for other inequalities and equalities involving functions.

Should each f_n be *continuous* on I, the sets in curly brackets on the right-hand side of equation (2) will all be closed in I, hence so will each set $F_N(\varepsilon)$. Thus each $C(\varepsilon)$, while not necessarily itself closed, will be a *countable union* of closed sets (an "F_σ-set" in the now-standard terminology introduced in 1899 by Baire [**Ba**]).

RETURNING NOW to the situation described by equation (1), we have, from the fact that $(f_n(x))$ is, for each $x \in I$, a Cauchy sequence of real numbers:

$$I = C(\varepsilon) = \bigcup_{N\in\mathbb{N}} F_N(\varepsilon) \qquad \forall\, \varepsilon > 0\,. \tag{3}$$

Since each $F_n(\varepsilon)$ is a closed subset of I, this sets the stage for:

The Baire Category Theorem ([**Ba**],1899)). *If I is a countable union of closed subsets, then at least one of these must contain an interval.*

We'll defer the proof of this famous result to §5, after we've come to appreciate it's role in unravelling the secrets of Baire-Class 1.

Corollary 3. *Suppose f is a function of Baire-class 1, as described by equation (1) above. Then for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ and an open interval $J = J(\varepsilon)$ such that for every $x \in J$:*

(a) $|f_n(x) - f_m(x)| \leq \varepsilon$ *for each* $n, m \geq N$, *and*

(b) $|f_N(x) - f(x)| \leq \varepsilon$ *for every* $n \geq N$.

Proof. Part (a) follows directly from the Baire Category Theorem and the representation (3) of I as a countable union of closed sets. Part (b) follows from part (a) upon setting $m = N$ and letting $n \to \infty$. □

SUMMARY for a Baire-Class 1 function as described by equation (1):

> *For every* $\varepsilon > 0$ *there exists* $N = N(\varepsilon) \in \mathbb{N}$ *such that* $|f - f_N| < \varepsilon$ *on some open interval* $J(\varepsilon)$ *of* I.

Our next task will be to understand what degree of continuity (if any) the "ε-uniform approximator" f_N passes on to its "approximatee" f. Here "degree of continuity" will be measured by the notion of "oscillation."

3 Oscillation

Suppose S is any set and $f : S \to \mathbb{R}$. The *oscillation* of f over S is defined as

$$\Omega_f(S) = \sup_{x,y \in S} |f(x) - f(y)| . \tag{4}$$

Note that $\Omega_f(S)$ decreases with S, i.e.,

$$S_1 \supset S_2 \implies \Omega_f(S_1) \geq \Omega_f(S_2) .$$

It follows that: for an open interval J a function $f : J \to \mathbb{R}$, and a point $x \in J$, the limit

$$\omega_f(x) = \lim_{r \to 0+} \Omega_f(J(x, r)) \tag{5}$$

Here $J(x, r)$ is the open interval $(x - r, x + r)$.

exists and is ≥ 0. We call $\omega_f(x)$ the *oscillation of* f *at* x. Here are some of its crucial properties; the proofs of which are all easy exercises.

Proposition 4. *For a real-valued function* f *on an open interval* J:

(a) *If* $|f| \leq M$ *on* J *then* $\omega_f \leq 2M$ *on* J.

(b) f *is continuous at a point* $x \in J$ *if and only if* $\omega_f(x) = 0$.

(c) *For each* $\lambda > 0$, *the set* $\{\omega_f < \lambda\}$ *is open in* J.

(d) *If both* f *and* g *are real-valued functions on* J, *then at each point of* J

$$\omega_{f+g} \leq \omega_f + \omega_g .$$

Corollary 5. *Suppose:*

- *J is an open real interval,*
- *$g\colon J \to \mathbb{R}$ is a function that is* continuous *on J,*
- *ε is a positive real number, and*
- *f is any real-valued function on J with $|f(x) - g(x)| \le \varepsilon$ for each $x \in J$.*

Then: $\omega_f(x) \le 2\varepsilon$ for every $x \in J$.

Proof. Fix $x \in J$. Then from Proposition 4(d):

$$\omega_f(x) = \omega_{(f-g)+g}(x) \le \omega_{f-g}(x) + \omega_g(x).$$

Now $\omega_g(x) = 0$ from the continuity of g and Proposition 4(b), while by part (a) of that Proposition and the uniform estimate on $|f - g|$ we have $\omega_{f-g}(x) \le 2\varepsilon$. □

4 *Proof of Baire's Continuity Theorem*

Recall that we're given a closed interval I of the real line and a real-valued function f on I that's the pointwise limit of a sequence (f_n) of functions that are continuous on I.

To show: f is continuous on a dense subset of I.

To this end, fix $\varepsilon > 0$ and let $G_\varepsilon = \{x \in I\colon \omega_f(x) < \varepsilon\}$. By part (c) of Proposition 4 G_ε *is open in* I.

Claim: G_ε *is dense in* I.

Indeed: we know from Corollary 3 that there is an interval $J \subset I$ an an index $N \in \mathbb{N}$ such that $|f - f_N|$ is $\le \varepsilon/3$ on J. Thus ω_f is $\le 2\varepsilon$ at every point of J, hence $J \subset G_\varepsilon$. In particular, this shows (finally!) that G_ε is non-empty.

However the Claim demands that we show G_ε has non-empty intersection with *every* open subinterval H of I, not just J (which it contains). No problem: apply the result of the last paragraph, with $\overline{H}$ (the closure of H) in place of I, and $G_\varepsilon \cap H$ (which is open and dense in $\overline{H}$) in place of G_ε. The result is that $G_\varepsilon \cap H$ is not empty, as desired.

Now the set of points of continuity of f (indeed, for any function on I) is

$$\{\omega_f = 0\} = \bigcap_{n\in\mathbb{N}} \{\omega_f < 1/n\} = \bigcap_n G_{1/n}, \tag{6}$$

a countable intersection of dense, open subsets of I which, by an "alternate version" of the Baire Category Theorem (see next section) is dense. □

IN ADDITION TO his designation "F_σ" for countable unions of closed sets, Baire introduce the term "G_δ" for countable *intersections* of *open* sets. Thus Eqn. (6) shows that for any real-valued function f on I:

> *The set of points of continuity is a G_δ set.*

...which, in this generality, may be empty.

5 Proof of Baire's Category Theorem

WE'VE EMPLOYED two equivalent versions of the Baire Category Theorem (henceforth, the "BCT"), which we'll attack in its natural setting: that of metric spaces (X, d).[4]

[4] Feel free to restrict attention to the metric space most comfortable for you, e.g. $X = \mathbb{R}$ or some closed interval therein, and $d =$ the usual length function on $\mathbb{R}$

The Original BCT. *If a complete metric space is the countable union of closed, then at least one of these sets has nonvoid interior.*

The Alternate BCT. *In a complete metric, every countable intersection of dense open sets is dense.*

WE'LL PROVE the equivalence of these two versions, after which we'll establish the "alternate" one.

Proof that "Original BCT" implies "Alternate BCT." Assuming the Original BCT, suppose $\{G_n\}$ is a countable family of dense, open subsets of our metric space X.

TO SHOW: $G = \bigcap_n G_n$ is dense, i.e., G has nontrivial intersection with every nonvoid open subset of X.

Since each G_n is open, its complement $F_n = X \backslash G_n$ is closed, and since G_n is dense, F_n is nowhere dense, i.e., dense in no open set.[5] By the Original BCT, $X \neq \bigcup_n F_n = X \backslash \bigcap_n G_n$. Thus $G = \bigcap_n G_n$ is not empty.

[5] If F_n were dense in some open subset V of X, then since F_n is closed, V would be contained in F_n. But then no point of V could be a limit point of G_n, hence G_n could not be dense.

We're now in a situation similar to the one encountered in our proof of Baire's Continuity Theorem. We want to show that G intersects every nonvoid open set, but have just shown that it's nonempty! Once again, generality saves the day!

Let V be a nonvoid open subset of X, and replace X by $\overline{V}$, the closure of V in X. Then $\overline{V}$ is a complete metric space (in the metric of X), in which each $G_n \cap V$ is a dense open subset. It follows from the last paragraph that $G \cap V = \bigcap_n (G_n \cap V)$ is nonempty, i.e., that G intersects V. □

Proof that "Alternate BCT" implies "Original BCT." Suppose now that we've prove the alternate BCT. Suppose $\{F_n\}$ is a countable collection of closed, subsets of X with $X = \bigcup_n F_n$. Then $G_n = X \backslash F_n$ is open in X for each $n \in \mathbb{N}$, and $\bigcap_n G_n$ is empty. By the alternate form of the BCT, some G_n some G_n must fail to be dense, hence the corresponding complement F_n must have nonvoid interior (i.e., it must contain a nonvoid open set. □

BEFORE PROVING THE ALTERNATE BCT, let's take a moment to appreciate the special nature of dense open sets. We know that two dense sets may be disjoint (e.g., the rationals and the irrationals in the real line), and the same can happen for two open sets (e.g. the intervals $(0,1)$ and $(1,2)$ of $\mathbb{R}$). But this can't happen for two sets that are *both dense and open*.

Indeed, in a metric space, suppose V_1 and V_2 are both dense and open, and let U be any nonvoid open subset of our space. Since V_1 is dense, it has a nonvioid intersection with U. Since $V_1 \cap U$ is open and non-void, and V_2 is dense, *it* has a non-void intersection with $V_1 \cap U$. Thus $V_1 \cap V_2$ has non-void intersection with any nonvoid open set U, and is therefore dense in our space.

More generally, an easy induction shows that:

> *In any metric space, the intersection of a finite family of dense open sets is dense.*

THE POINT OF the "Alternate BCT" is that for *complete* metric spaces this result extends to *countable* families of dense open sets. The argument will be similar to the one you just used to show that intersections of finite collections of dense open sets are dense—but now it requires a little more precision

EXERCISE. In an *incomplete* metric space this result can fail.

Proof of the Alternate BCT. Our setting is a metric space (X, d) that is *complete*, i.e., in which every Cauchy sequence converges.

Let's fix a countable collection $\{V_n\}$ of subsets of X, each of which is open and dense in X, and set $G = \bigcap_n V_n$.

TO SHOW: *G is dense in X.*

Since each open set V_n is dense in X, they're all non-empty. Since V_1 is dense, it has a non-void open intersection with V_2. Fix any open ball B_1 whose closure lies in V_1. so we've found a ball B_1 in $V_1 \cap V_2$ whose closure also lies in that intersection. Since $B_1 \cap V_2$ is open (thanks to the open-ness of B_1 and V_1, and nonempty (thanks to density of V_1), we can find an open ball B_2 whose closure $\overline{B_2}$ lies in $B_1 \cap V_2$.

EXERCISE. There are metric spaces for which closure of an open ball, say of radius r and center x_0, need not coincide with the set of points whose distance from x_0 is $\leq r$ (although the closure of that open ball does lie in this latter "closed ball").

CONTINUING IN THIS MANNER (i.e., by induction) we can find, for each index n, an open ball B_n whose closure $\overline{B_n}$ lies in $B_{n-1} \cap V_n$. Thanks to *Cantor's Intersection Theorem* (see below), the completeness of X now guarantees that $\bigcap_n \overline{B_n}$, hence its superset $\bigcap_n V_n$, is nonempty.

Now we're in a familiar situation. Our goal is to prove that G is dense in X, but so far we just know it's non-empty. To complete the proof, fix a nonempty open subset U of X and apply the result obtained in the last paragraph with X replaced by the closure $\overline{U}$ of U and V_n replaced by $V_n \cap U$. The result—thanks to the fact that every closed subset of a complete metric space is itself complete (in the original metric)—is that $G \cap U = \bigcap_n (V_n \cap U)$ is nonempty, which establishes the density of G in X. □

CANTOR'S INTERSECTION THEOREM. *In a complete metric space, every countable decreasing sequence of non-empty closed sets has non-empty intersection.*

To prove this, note that, in the argument above, you may choose the radius of B_n to be $\leq 1/2\times$ the radius of B_{n-1}. Letting x_n denote the center of B_n, a little argument using the triangle inequality and the Geometric Series Theorem shows that (x_n) is a Cauchy sequence in X, which by completeness must converge to some $x_0 \in X$. By construction, the $\overline{B_n}$'s form a decreasing sequence of closed sets, the n-th one of which contains $\{x_n, x_{n+1}, \ldots\}$, hence also x_0. Thus x_0 lies in $\bigcap_n \overline{B_n}$, hence in its superset $G = \bigcap V_n$.

6 *Concluding remarks*

6.1 *Baire spaces.*

A *Baire space* is a topological space for which the intersection of every countable family of dense, open sets is dense. Thus: *Every complete metric space is a Baire space.*

Although completeness is not preserved by homeomorphisms, "Baire-ness" *is* preserved. Thus:

EXAMPLE: the open unit interval, which is not complete in the metric of the real line, is nonetheless homeomorphic to the line

> *Every topological space homeomorphic to a complete metric space is a Baire space.*

In particular:

> *The conclusion of the Baire Category Theorem holds for every open interval of the real line.*

Note also that our proof of the Baire Category Theorem depended only upon the fact that in a complete metric space, every decreasing sequence of nonempty closed sets has nonempty intersection. Thus:

> *Every compact topological space is a Baire space.*

6.2 *Dense G_δ sets are (usually) uncountable!*

For definiteness, let's take as our initial setting the real line. Suppose G is a dense G_δ subset of $\mathbb{R}$. Thus $G = \bigcap_n V_n$ where each V_n is an *open* subset that is itself dense in $\mathbb{R}$. To see that G is uncountable, suppose—for the sake of contradiction—that it is not, i.e., suppose that $G = \{x_1, x_2, \ldots\}$. Then $W_n = V_n \backslash \{x_1, x_2, \ldots x_n\}$ is open and dense in $\mathbb{R}$, and

$$\bigcap_n W_n = G \backslash \{x_n\}_1^\infty = G \backslash G = \emptyset,$$

contrary to the Baire Category Theorem. □

The argument above works, word-for-word, to establish:

Proposition. *In any Baire space without isolated points, every dense G_δ set is uncountable.*

6.3 *Back to the beginning.*

Let's return to our original closed interval I of the real line. We've shown that for every real-valued function on I, its set of points of continuity is a (possibly empty) G_δ. However, if our function is of Baire-Class 1 we've shown that its G_δ-set of continuity is *dense*. In view of what we've just seen in §6.2, we now know that this set is, in addition, *uncountable*.

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Here you'll find not only Darboux's IVP theorem for derivatives, but also as his introduction of the "Darboux integral" (based on the limit of upper and lower integrals).

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