The Riemann Hypothesis

it's all about zero

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The Riemann Hypothesis (1859)

"All the nontrivial zeros of the Riemann zeta function lie on the critical line."

Bernhard Riemann
1826-1866

The Riemann zeta function (1859)

Definition. $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\mathrm{Re}\, s > 1)$

Properties. For $\mathrm{Re}\, s > 1$:

* Series defining ζ converges *absolutely*
* Series converges *uniformly* on $\{\mathrm{Re}\, s \geq \alpha\}, \quad \forall\ \alpha > 1$.
* ζ is analytic on $\{\mathrm{Re}\, s > 1\}$.

The Euler Product Formula

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

Euler Formula Consequences

(a) $\#$ primes $= \infty$

(b) $\sum_p \frac{1}{p} = \infty$,

(c) ζ non-vanishing on $\{\mathrm{Re}\, s > 1\}$.

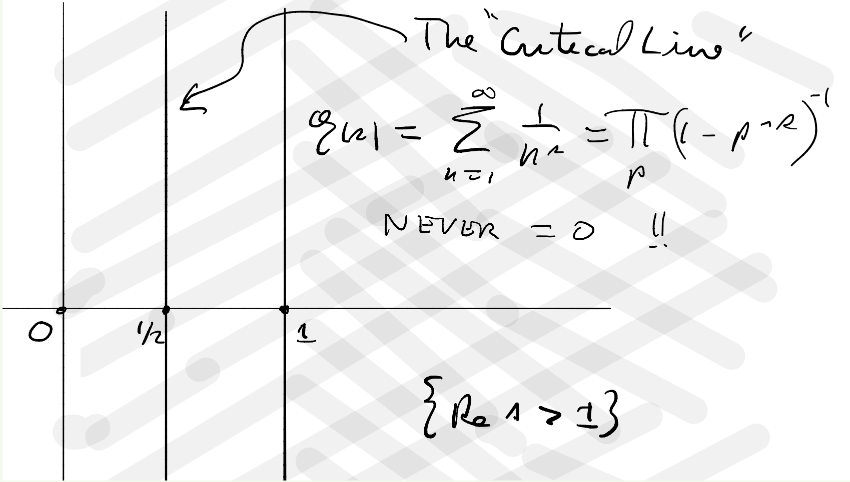
Theorem. *ζ continues analytically to $\{\mathrm{Re}\, s > 0\}$ w/ simple pole at $s = 1$.*

Proof. Euler-Maclaurin Summation:

$$\zeta(s) = \frac{s}{s-1} + \underbrace{\int_1^{\infty} \{t\}\, t^{-s-1}\, dt}_{\text{analyt. in } \{\mathrm{Re\,s}>0\}}$$

The Riemann Hypothesis

"All nontrivial zeros of $\zeta(s)$ lie on the critical line."



The "completed" zeta function (Riemann 1859)

Definition. For $\operatorname{Re} s > 0$:

$$\xi(s) = s(1-s)\,\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$$

The Gamma Function $(\operatorname{Re} s > 0)$

$$\Gamma(s) = \int_{t=0}^{\infty} e^{-t}\,t^{s-1}\,dt$$

Some Γ-Properties

(a) Γ *analytic on* $\{\operatorname{Re} s > 0\}$.

(b) $\Gamma(s+1) = s\,\Gamma(s) \qquad (\operatorname{Re} s > 0)$.

(c) Γ *extends analytically to* $\mathbb{C}\backslash\{0, -1, -2, \ldots\}$ *with simple poles at* $\{0, -1, -2, \ldots\}$.

Proof of (c). From (b):

$$\Gamma(s) = \frac{\Gamma(s+1)}{s} = \frac{\Gamma(s+2)}{(s+1)s} = \frac{\Gamma(s+3)}{(s+2)(s+1)s} = \cdots$$

Riemann's Rep'n Thm. $(\operatorname{Re} s > 0)$

$$\xi(s) = \int_{t=1}^{\infty} [t^{s/2} + t^{(1-s)/2}]\,\Phi(t)\,\frac{dt}{t} \;-\; 1$$

Corollary

(a) $\xi(s)$ extends to an entire function, and

(b) $\xi(s) = \xi(1-s)$ ("Riemann's fnl eqn")

Trivial zeros

Γ-Properties

(a) Γ *analytic on* $\{\mathrm{Re}\, s > 0\}$.

(b) $\Gamma(s+1) = s\,\Gamma(s) \qquad (\mathrm{Re}\, s > 0)$.

(c) Γ *extends analytically to* $\mathbb{C}\backslash\{0, -1, -2, \ldots\}$ *with simple poles at* $\{0, -1, -2, \ldots\}$.

(d) $\Gamma(s)$ never $= 0$.

(e) $1/\Gamma$ is entire (i.e., analytic on $\mathbb{C}$)

ξ-Properties

(a) $\xi(s)$ extends to an entire function for which

(b) $\xi(s) = \xi(1-s)$.

The completed zeta function

$$\xi(s) := s(1-s)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

Solve for $\boldsymbol{\zeta(s)}$:

$$(1-s)\zeta(s) = \frac{1}{s\Gamma(s/2)}\,\pi^{s/2}\,\xi(s)$$

Theorem. The zeta function $\zeta(s)$ extends analytically to $\mathbb{C}\backslash\{1\}$, with:

(a) A simple pole at $s = 1$, and

(b) simple zeros at $s = -2, -4, \ldots$ (the "trivial" zeros).

Corollary. The *nontrivial* zeros of $\zeta(s)$ are *precisely* the zeros of $\xi(s)$.

The "critical strip"

Recall: "Completed zeta" is entire.

$$\xi(s) = s(1-s)\,\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$$

Theorem. *The non-trivial zeros of ζ lie in the "critical strip" $\{0 < \operatorname{Re} s < 1\}$.*

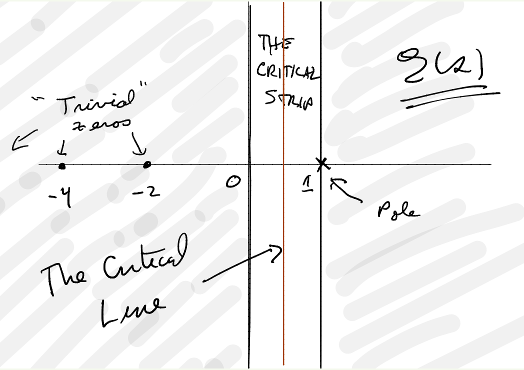
Proof. From Euler Product:

$$\operatorname{Re} s > 1 \implies \zeta(s) \neq 0$$
$$\implies \xi(s) \neq 0$$

Since $\xi(s) = \xi(1-s)$, have also:

$$\operatorname{Re} s < 0 \implies \xi(s) \neq 0$$

Conclude: *The non-trivial zeros of ζ lie in the <u>closure</u> of the critical strip.*



To do!

(a) ζ nonzero on the line $\{\operatorname{Re} s = 1\}$, on $\{\operatorname{Re} s = 0\}$. (Crucial for prime number theorem!)

(b) Do there *exist* any nontrivial zeros?

Some References

- H. M. Edwards, *Riemann's Zeta Function*, Academic Press 1974.
- Leonhard Euler, *Variae observationes circa series infinitas*, St. Petersburg Academy, 1737. Downloadable from the "1737" link at https://bit.ly/2IQlWOg
- Bernhard Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte der Berliner Akademie. In *Gesammelte Werke*, Teubner, Leipzig (1892), Reprinted by Dover, New York (1953).

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  English translation by D.R. Wilkens, downloadable here: https://bit.ly/31pQxsp