The Riemann Hypothesis
the “nontrivial” zeros

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The Riemann Hypothesis (1859)

"All the nontrivial zeros of the Riemann zeta function lie on the critical line."

Bernhard Riemann
1826-1866

The Riemann zeta function (1859)

Definition. $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\mathrm{Re}\, s > 1)$

Properties. For $\mathrm{Re}\, s > 1$:

* Series defining ζ converges *absolutely*
* Series converges *uniformly* on $\{\mathrm{Re}\, s \geq \alpha\}, \quad \forall\, \alpha > 1$.
* ζ is analytic on $\{\mathrm{Re}\, s > 1\}$.

The Euler Product Formula

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

Euler Formula Consequences

(a) $\#$ primes $= \infty$

(b) $\sum_p \frac{1}{p} = \infty$,

(c) ζ non-vanishing on $\{\mathrm{Re}\, s > 1\}$.

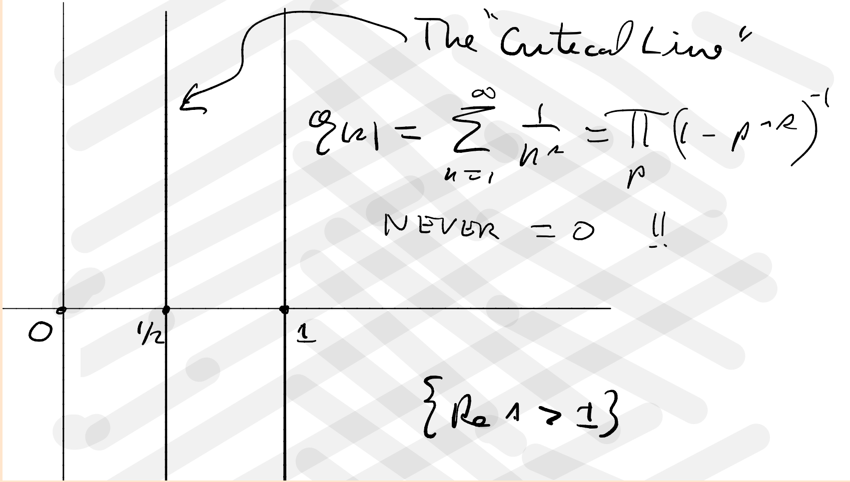
Theorem. *ζ continues analytically to $\{\mathrm{Re}\, s > 0\}$ w/ simple pole at $s = 1$.*

Proof. Euler-Maclaurin Summation:

$$\zeta(s) = \frac{s}{s-1} + \underbrace{\int_1^{\infty} \{t\}\, t^{-s-1}\, dt}_{\text{analyt. in } \{\mathrm{Re\, s} > 0\}}$$

The Riemann Hypothesis

"All nontrivial zeros of $\zeta(s)$ lie on the critical line."



The "completed" zeta function (Riemann 1859)

Definition. For $\mathrm{Re}\, s > 0$:

$$\xi(s) = s(1-s)\,\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$$

The Gamma Function $(\mathrm{Re}\, s > 0)$

$$\Gamma(s) = \int_{t=0}^{\infty} e^{-t}\, t^{s-1}\, dt$$

Some $\boldsymbol{\Gamma}$-Properties

(a) Γ *analytic on* $\{\mathrm{Re}\, s > 0\}$.

(b) $\Gamma(s+1) = s\,\Gamma(s) \qquad (\mathrm{Re}\, s > 0)$.

(c) Γ *extends analytically to* $\mathbb{C}\backslash\{0,-1,-2,\ldots\}$ *with simple poles at* $\{0,-1,-2,\ldots\}$.

(d) $\Gamma(s)$ *is never* 0.

Proof of (c). From (b):

$$\Gamma(s) = \frac{\Gamma(s+1)}{s} = \frac{\Gamma(s+2)}{(s+1)s} = \frac{\Gamma(s+3)}{(s+2)(s+1)s} = \cdots$$

Riemann's Rep'n Thm. $(\mathrm{Re}\, s > 0)$

$$\xi(s) = \int_{t=1}^{\infty} [t^{s/2} + t^{(1-s)/2}]\,\Phi(t)\,\frac{dt}{t} \;-\; 1$$

Corollary

(a) $\xi(s) = \xi(1-s)$ ("Riemann's fnl eqn")

(b) $\xi(s)$ extends to an entire function.

(c) $\xi(s) \neq 0$ outside $\{0 \le \mathrm{Re}\, s \le 1\}$.

Zeros: Trivial & Nontrivial; The Critical Strip

Γ-Properties

(a) Γ *analytic on* $\{\mathrm{Re}\, s > 0\}$.

(b) $\Gamma(s+1) = s\,\Gamma(s) \qquad (\mathrm{Re}\, s > 0)$.

(c) Γ *extends analytically to* $\mathbb{C}\backslash\{0, -1, -2, \ldots\}$ *with simple poles at* $\{0, -1, -2, \ldots\}$.

(d) $\Gamma(s)$ never $= 0$.

(e) $1/\Gamma$ is entire (i.e., analytic on $\mathbb{C}$)

ξ-Properties

(a) $\xi(s)$ extends to an entire function for which

(b) $\xi(s) = \xi(1-s)$.

The completed zeta function

$$\xi(s) := s(1-s)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

$$\therefore\ (1-s)\zeta(s) = \frac{1}{s\Gamma(s/2)}\,\pi^{s/2}\,\xi(s) \quad (*)$$

Theorem. The zeta function $\zeta(s)$ extends analytically to $\mathbb{C}\backslash\{1\}$, with:

(a) A simple pole at $s = 1$, and

(b) simple zeros at $s = -2, -4, \ldots$ (the "trivial" zeros).

Corollary. The *nontrivial* zeros of $\zeta(s)$ are *precisely* the zeros of $\xi(s)$; these all lie in the "critical strip" $\{0 < \mathrm{Re}\, s < 1\}$.

(*) $\frac{1}{2}$Vol. of n-ball $= \frac{\pi^{n/2}}{n\Gamma(n/2)}$ (h/t: Sheldon)

$\exists$? nontrivial zeros

The completed zeta function

$$\xi(s) \ := \ s(1-s)\,\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$$

Riemann's Integral Representation.
For all $s \in \mathbb{C}$:

$$\xi(s) = \int_{t=1}^{\infty} [t^{s/2} + t^{(1-s)/2}]\Phi(t)\frac{dt}{t} - 1$$

Corollary. $\xi(s)$ *is an entire function of:*

(a) *Order 1*: $|\xi(s)| = \mathrm{O}(e^{|s|^\rho}) \ \ \forall \rho > 1$
but not for <u>any</u> $\rho < 1$.

(b) *Type* $= \infty$:
$\nexists\ A > 0 : |\xi(s)| = \mathrm{O}(e^{A|s|})$

Theorem. *Every entire function of order 1 and infinite type has infinitely many zeros.*

Proof. Next slide

Corollary. $\xi(s)$ *has infinitely many zeros.*

Corollary. $\zeta(s)$ *has infinitely many zeros in the critical strip.*

Remark. Theorem false for "finite type".

Example. $F(s) = e^{as}$ for any $a \in \mathbb{C}$

Remark. Theorem holds for all entire functions of fractional order and/or infinite type.

Order, Type, Zeros

To Show: $\xi(s)$ has infinitely many zeros.

Consequence: $\zeta(s)$ has infinitely many non-trivial zeros, all in the "critical strip".

Know: $\xi(s)$ is entire, order 1, type ∞.

To Prove: Any such entire function has infinitely many zeros.

Polynomial analogy: Zeros $\Longleftrightarrow$ Growth
"Partially true" for entire functions!

Suppose: F entire, order 1,
with just finitely many zeros.

To Show: F must be of finite type, i.e.,

$$F(z) = \mathrm{O}(e^{a|z|})$$

Know: $\exists$ *polynomial* P *such that*
$F(z)/P(z)$ *is entire, with* *<u>no</u>* *zeros.*

$\therefore$ $\exists$ entire $Q(z)$ such that
$$F(z) = P(z)\, e^{Q(z)}.$$

$\therefore$ e^Q of order 1.

Claim: $Q(z) = Az + B$

$\therefore\ F(z) = P(z)e^{Q(z)} = \mathrm{O}(e^{C|z|})\ \forall\, C > |A|$

i.e., F is of finite type. $\square$

Proving the Claim

The Claim: For Q entire: if e^Q of order 1, then $Q(z) = Az + B$.

Note: e^Q order 1 $\Rightarrow$

$$\operatorname{Re} Q(z) = \mathrm{O}(|z|^\rho) \quad \forall\ \rho > 1.$$

Suppose u harmonic on $\mathbb{C}$.

Liouville's Theorem. If $|u(re^{i\theta})| = \mathrm{O}(r^\rho)$ for some $\rho > 0$, then u is a (harmonic) polynomial of degree $\leq \rho$.

Proof. $$u(re^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n\, r^{|n|}\, e^{in\theta}\ .$$

$$\therefore \quad a_n\, r^{|n|} = \frac{1}{2\pi}\int_0^{2\pi} u(r\,e^{i\theta})e^{-in\theta}\, d\theta$$

$$\therefore \quad |a_n|\, r^{|n|} \leq \frac{1}{2\pi}\int_0^{2\pi} |u(r\,e^{i\theta})|\, d\theta \leq \text{const.}\, r^\rho.$$

Conclude: if $|n| > \rho$ then

$$|a_n| \leq \text{const.}\, r^{\rho-|n|} \to 0 \text{ as } r\to\infty. \qquad \square$$

To do

(a) Prove Riemann's Integral Rep'n:

$$\xi(s) = \int_{t=1}^{\infty} [t^{s/2} + t^{(1-s)/2}]\Phi(t)\frac{dt}{t} - 1$$

(b) Study the *Prime Number Theorem* (needs: $\operatorname{Re} s = 1 \Rightarrow \zeta(s) \neq 0$)

Some References

- H. M. Edwards, *Riemann's Zeta Function*, Academic Press 1974.
- Leonhard Euler, *Variae observationes circa series infinitas*, St. Petersburg Academy, 1737. Downloadable from the "1737" link at https://bit.ly/2IQlWOg
- Bernhard Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte der Berliner Akademie. In *Gesammelte Werke*, Teubner, Leipzig (1892), Reprinted by Dover, New York (1953).

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  English translation by D.R. Wilkens, downloadable here: https://bit.ly/31pQxsp