The Basel Problem

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The Basel Problem

- *Pietro Mengoli* (1644): $\sum_{n=1}^{\infty} \frac{1}{n^2} = ?$
- *Johan Bernoulli* (1689): Popularized Mengoli's Question.
- *Leonhard Euler* (1735–1741):
  - $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$
  - Evaluated $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ for $k = 2, 3, 4, \ldots$.

Analyticity—Easy stuff

Setting: Ω an open subset of $\mathbb{C}$.

Definitions: To say $f : \Omega \to \mathbb{C}$ is:

- *Complex-differentiable* at $z_0 \in \Omega$ means

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists} \quad (\in \mathbb{C})$$

- *Analytic* on Ω means it's complex-diff'ble at each $z_0 \in \Omega$.

Consequences:

- Complex diff'n obeys *all the usual rules* of calculus (same proofs)
- Sums & products of analytic functions are analytic, as are quotients and compositions—when they make sense.

Analyticity—Amazing stuff!

Assume: f analytic on Ω. Then:

- f' is analytic on Ω (!!)
- Hence $f^{(n)}$ is analytic on Ω for each $n \in \mathbb{N}$.
- For each $z_0 \in \Omega$, the Taylor series for f at z_0 *converges to f in the largest possible open disc centered at* z_0:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \quad (|z - z_0| < \text{dist. from } z_0 \text{ to } \partial\Omega)$$

Example: *The complex exponential.*

$$e^z := e^x \underbrace{(\cos y + i \sin y)}_{:= e^{iy}} \qquad (z = x + iy; x, y \in \mathbb{R})$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} z^n \qquad (\text{analytic in } \mathbb{C})$$

Exponential Escapades

Properties of the complex exponential:

- $e^z := e^x \underbrace{(\cos y + i \sin y)}_{:= e^{iy}} \quad (z = x + iy \in \mathbb{C}).$

- $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \qquad (z \in \mathbb{C})$

- Analytic on $\mathbb{C}$: $\frac{d}{dz} e^z = e^z \qquad (z \in \mathbb{C}).$

- $2\pi i$-Periodic: $e^z = e^{2\pi i z} \quad (z \in \mathbb{C}).$

- $e^z = 1 \iff z \in 2\pi i \mathbb{Z}.$

Example. $F(z) := \frac{e^z - 1}{z} = \frac{z + z^2/2! + z^3/3! + \ldots}{z} = 1 + z/2 + z^2/3! + \ldots$

Conclude: F is analytic on $\mathbb{C}$ (if we define $F(1) = 1$).

The "Bernoulli Function":

$$B(z) \;:=\; \frac{1}{F(z)} \;=\; \begin{cases} \frac{z}{e^z - 1} & \text{if } z \in \mathbb{C}\backslash\{1\} \\ 1 & \text{if } z = 1 \end{cases}$$

$$= \sum_{n=0}^{\infty} \frac{b_n}{n!} z^n \qquad \text{(the } b_n\text{'s are the \textit{Bernoulli numbers})}$$

$$= 1 - \frac{1}{2} z + \frac{1}{12} z^2 - \frac{1}{720} z^4 + \text{ even powers of } z$$

Conclude:

- The Bernoulli function is analytic in $(\mathbb{C}\backslash 2\pi i\mathbb{Z}) \cup \{0\}$.
- Its "Bernoulli series" converges for $|z| < 2\pi$.

Basel & Bernoulli: Part 1

A modified Bernoulli Function

$$H(z) \;:=\; \frac{2\pi i}{e^{2\pi i z}-1} \;=\; \frac{B(2\pi i\, z)}{z} \qquad (z \in \mathbb{C}\backslash\mathbb{Z})$$

- H is analytic in $\mathbb{C}\backslash\mathbb{Z}$, and for $0 < |z| < 1$:
- $H(z) = \frac{1}{z} - \pi i - \frac{\pi^2}{3} z - \frac{\pi^4}{45} z^3 + \cdots$ (odd powers of z, neg. coeffs)

Conclude: For $n \in \mathbb{Z}$ and $0 < |z - n| < 1$

- $H(z) = H(z-n) = \frac{1}{z-n} - \pi i - \frac{\pi^2}{3}(z-n) - \frac{\pi^4}{45}(z-n)^3 + \cdots$

In short:

- H has a *pole of order 1* at each $n \in \mathbb{Z}$, and
- Its *Laurent Series* with center n converges in the "punctured" open disc of radius 1, center at n.

Laurent Series

Theorem. *If f is analytic in a "punctured disc" centered at $z_0 \in \mathbb{C}$, of radius r. Then $\exists$ unique coefficient "sequence" $(a_n)_{n\in\mathbb{Z}}$ such that*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z - z_0)^n \qquad (0 < |z - z_0| < r).$$

Definition. a_{-1} is the *residue* of f at z_0 (notation: $\operatorname{res}(f, z_0)$).

Example (*A modified Bernoulli function*).

$$H(z) := \frac{2\pi i}{e^{2\pi i z} - 1} = \frac{B(2\pi i\, z)}{z} \qquad (z \in \mathbb{C}\backslash\mathbb{Z})$$

- H is analytic in $\mathbb{C}\backslash\mathbb{Z}$, and for $0 < |z| < 1$:
- $H(z) = \frac{1}{z} - \pi i - \frac{\pi^2}{3} z - \frac{\pi^4}{45} z^3 + \cdots$ (odd powers of z, neg. coeffs)

Conclude: For $n \in \mathbb{N}$ and $0 < |z - n| < 1$

- $H(z) = H(z - n) = \frac{1}{z - n} - \pi i - \frac{\pi^2}{3}(z - n) - \frac{\pi^4}{45}(z - n)^3 + \cdots$
- $\operatorname{res}(H, n) = 1 \qquad (n \in \mathbb{Z})$

The Residue Theorem

Some hypotheses.

- Γ: a simple, closed, "sufficiently smooth" curve in the plane,
- Ω: the open region containing Γ and its "interior".
- S: a finite set of points in Ω
- f: a function analytic on $\Omega \backslash S$.

The Residue Theorem.

$$\textit{Hypotheses above} \Rightarrow \int_\Gamma f(z)\,dz = 2\pi i \sum_{s \in S} \operatorname{Res}(f, s)$$

Corollary (*The case* $S = \emptyset$).

$$f \text{ analytic in } \Omega \Rightarrow \int_\Gamma f(z)\,dz = 0.$$

This is a special case of *The Cauchy Integral Theorem*.

Residues and The Basel Problem I

Proposition. *If $n \in \mathbb{Z}$ and f is analytic in $\{|z-n|<1\}$, then*

$$f \text{ analytic in } \{|z-n|<1\} \Rightarrow \operatorname{Res}(f\,H, n) = f(n)$$

where the series on the right converges for $0 < |z-n| < 1$.

Proof. $f(z) = f(n) + (z-n) \times$ a power series in $(z-n)$

$$H(z) = \frac{1}{z-n} + \text{a power series in } (z-n)$$

$$\therefore \quad f(z)H(z) = \frac{f(n)}{z-n} + \text{a power series in } (z-n). \quad \text{QED}$$

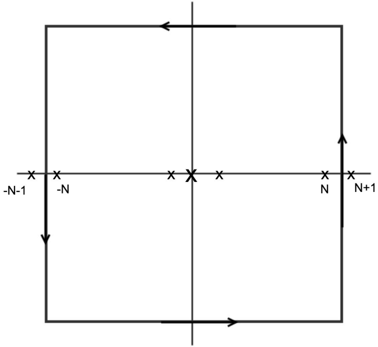
Corollary. *For each $n \in \mathbb{Z}\backslash\{0\}$, and $k \in \mathbb{N}$:*

$$\operatorname{Res}\left(\frac{H(z)}{z^{2k}}, n\right) = \frac{1}{n^{2k}}$$

Residues and the Basel Problem II

Fix $k \in \mathbb{N}$. Set $F(z) = \frac{1}{z^{2k}} H(z)$, & let

$\Gamma_N =$ positively oriented boundary of the square shown below.



The Residue Theorem $\Rightarrow$

$$\frac{1}{2\pi i} \int_{\Gamma_N} F(z)\, dz = \sum_{|n| \leq N} \mathrm{Res}\,(F, n) = \mathrm{Res}\,(F, 0) + 2 \sum_{n=1}^{N} \frac{1}{n^{2k}}$$

Residues and the Basel Problem III

Residue Theorem $\Rightarrow$ $$\sum_{n=1}^{N} \frac{1}{n^{2k}} = \frac{1}{4\pi i} \int_{\Gamma_N} F(z)\, dz - \frac{1}{2} \operatorname{Res}(F, 0)$$

for each $N \in \mathbb{N}$, where $F(z) = H(z)/z^{2k}$.

Estimates:

- $|z| \leq N + \frac{1}{2}$ on Γ_N, and
- $|H| \leq$ some M on Γ_N ("easy").

$$\therefore \quad |F(z)| = \frac{|H(z)|}{|z|^{2k}} \leq \frac{M}{N^{2k}} \qquad (\text{on } \Gamma_N)$$

$$\therefore \quad \left| \int_{\Gamma_N} F(z)\, dz \right| \leq \int_{\Gamma_N} |F(z)|\, |dz| \leq \frac{M}{N^{2K}} \times \text{length of } \Gamma_N \to 0 \quad \text{as } n \to \infty$$

Conclusion: $$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = -\frac{1}{2} \operatorname{Res}\left(\frac{H(z)}{z^{2k}}, 0 \right)$$

Basel Problem: Solution

So far:

- $$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} \;=\; -\frac{1}{2}\,\mathrm{Res}\left(\frac{H(z)}{z^{2k}}, 0\right)$$
- $$H(z) := \frac{2\pi i}{e^{2\pi i z} - 1}$$

  $$= \frac{1}{z} - \pi i - \frac{\pi^2}{3} z - \frac{\pi^4}{45} z^3 + \cdots \text{ (odd powers of } z, \text{ neg. coeffs)}$$

Conclude:

- $\mathrm{Res}\left(\frac{H(z)}{z^2}, 0\right) = -\frac{\pi^2}{3}$ so $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.
- $\mathrm{Res}\left(\frac{H(z)}{z^4}, 0\right) = -\frac{\pi^4}{45}$ so $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{90}$.
- ...

Residue Theorem: Simple Case

Notation. $C_r(z_0)$ = circle $\{|z-z_0|=r\}$, oriented positively (CCW).

Proposition. $$\int_{C_r(z_0)} (z-z_0)^n \, dz = \begin{cases} 0 & \text{if } \; n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases} \qquad (n \in \mathbb{Z}).$$

Proof. Parameterize $C_r(z_0)$: $z = z_0 + re^{i\theta} \quad (0 \leq \theta \leq 2\pi)$.

$$\int_{C_r(z_0)} (z-z_0)^n \, dz = \int_0^{2\pi} r^n e^{in\theta} \underbrace{ire^{i\theta}\, d\theta}_{=dz} = i\, r^{n+1} \int_0^{2\pi} e^{i(n+1)\theta}\, d\theta$$

Theorem. If f is is analytic in punctured disc $0 < |z-z_0| < R$, $0 < r < R$, and $C = C_r(z_0)$, then:

$$\int_C f(z)\, dz = 2\pi i \, \mathrm{Res}\,(f, z_0)$$

Proof. $f(z) = \sum_{n\in\mathbb{Z}} a_n (z-z_0)^n \qquad (0 < |z-z_0| < R)$.

$$\therefore \quad \int_C f(z)\, dz = \sum_{n\in\mathbb{Z}} a_n \int_C (z-z_0)^n = 2\pi i\, a_{-1} \; .$$

Some References

- P. D. Lax and L. Zalcman, *Complex Proofs of Real Theorems* University Lecture Series #8, American Math. Society 2012.

  This talk based on Chapter 1.

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  A first graduate course in complex analysis

- M. Beck et al, *A First Course in Complex Variables* Online text downloadable from http://www.math.binghamton.edu/dennis/complex.pdf or http://math.sfsu.edu/beck/complex.html

  An undergraduate course in complex analysis aimed at math majors

- D. Sarason, *Complex Function Theory*, 2nd edition. American Math. Society 2007.

  An honors-undergraduate/graduate course in complex analysis