Banach Basis Basics II

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Review

Defn. A *basis* for a vector space V is a set $E \subset V$ such that each $v \in V$ is a unique linear combination of the vectors in E.

Rmk. "Unique represention of the zero-vector as a lin. comb. of vectors in E" $\equiv$ "E is *linearly independent.*"

AC $\implies$ every vector space has a basis.

Banach. In an infinite-dimensional Banach space, no basis can be countable.

Defn. A (Schauder) basis for a Banach space B is a *sequence* $E = (e_n)_1^\infty \subset B$ such that each $f \in B$ is a unique "extended" lin, comb, of the vectors in E, ... i.e.:

Given $f \in B$, $\exists!$ *scalar sequence* $(\alpha_n(f))$ *s.t.* $f = \sum_{n=1}^\infty \alpha_n(f) e_n$, *where the series converges for the norm of* B.

Examples (from last time). (a) Any basis in a finite dimensional vector space

(b) Any "orthonormal basis" for a separable Hilbert space. (h/t Parseval).

(c) Schauder's "tent basis" for $C([0,1])$.

(d) — from Seminar 12/6/2019.

For $n \in \mathbb{Z}$ and $f \in L^1([0,1]$, set
$\hat{f}(n) = \int_0^1 f(t) e^{-2\pi i n t}\, dt$.

Theorem (M. Riesz 1925?) If $1 < p < \infty$ and $f \in L^p$ then

$$\lim_{N \to \infty} \int \left| f(t) - \sum_{n=-N}^{N} \hat{f}(n) e^{int} \right|^p dt = 0$$

Cor: For $e_n(t) := e^{2\pi i n t}$, the seq. $(e_0, e_1, e_{-1}, e_2, e_{-2}, \ \ldots)$ is a Schauder basis for L^p $\quad (1 < p < \infty)$.

Banach's Projection Theorem

The basis $(e_n)_1^\infty$ for B provides, $\forall n \in \mathbb{N}$:

(a) *An n dim'l subspace* of B:

$$E_n = \text{span}\,(e_k)_{k=1}^n$$

(b) A *linear functional* $\alpha_n : B \to$ scalars

(c) A *linear projection* $P_n : B \to E_n$

$$P_n f = \sum_{k=1}^n \alpha_k(f) e_k \qquad (f \in B)$$

Question. Are the α_n's (equivalently: the P_n's) all *continuous* on B?

Answer (Banach 1932). YES

Proof (Banach). *Define* new norm on B:

$$|||f||| := \sup_n \|P_n(f)\| \quad (f \in B)$$

Easy. Basis functionals and projections are *continuous* for $|||\cdot|||$.

To Show: $|||\cdot|||$ & $\|\cdot\|$ induce *the same topology on* B.

Easy. $\|f\| \le |||f||| \quad (f \in B)$

Consequence: Continuity of

$$I : (B, |||\cdot|||) \longrightarrow (B, \|\cdot\|)$$

Banach: it's a homeomorphism.

Banach shows: $(B, |||\cdot|||)$ is *complete*!!

Then desired result follows upon applying the *Open Mapping Thm.* to I. $\square$

Corollary. *B can be re-normed so that $\|P_n\| = 1$ for each n.*

Basic sequences

Defn. To say a sequence in a Banach space is "basic" means: it is a basis for the closure of its linear span.

Thm. (Mazur 1935). *Every infinite dimensional Banach space contains a basic sequence.*

Enflo (1973). $\exists$ separable Banach spaces with *no* (Schauder) basis.

Mazur's Theorem says every separable Banach space has an *infinite dimensional subspace with a basis* !!

Thm. *Every subsequence of a basis is basic.*

Proof. Suppose (e_n) a basis for Banach space B.

For $n_k \nearrow \infty$ a subsequence of positive integers, write:

$$F = \text{closed linear span of the } e_{n_k}\text{'s}$$

TO SHOW: (e_{n_k}) is a basis for F.

For $g \in F$, we know: $g = \sum_n \alpha_n(g) e_n$

Enough to show: $\alpha_n(g) = 0 \ \ \forall n \notin (n_k)$

If $g \in \text{span}\,(e_{n_k})$ then obvious.

Otherwise, follows from this and the continuity of α_n. $\square$

Sub-Basis Examples

Recall: (a) The *complex exponentials:*

$$e_n(t) = e^{2\pi int} \quad (t \in \mathbb{R}, n \in \mathbb{Z})$$

(b) *M. Riesz* (1920's). If $1 < p < \infty$,

$$(e_0,\, e_1,\, e_{-1},\, e_2,\, e_{-2},\, \ldots)$$

is a basis for $L^p = L^p([0,1])$.

Defn. $H^p :=$ closed linear span in L^p of

$$E := (e_0,\, e_1,\, e_2,\, \ldots) \qquad (1 \le p < \infty)$$

The spaces H^p are called *Hardy spaces*.

M. Riesz. *E is a basis for H^p* $(1 < p < \infty)$

Riesz actually proved: the map $f \to \sum_{n\ge 0} \alpha_n(f) e_n$ is a *bounded projection* taking L^p onto H^p. i.e., H^p is a *complemented subspace* of L^p if $1 < p < \infty$.

The Hardy Spaces H^p $(1 < p < \infty)$.

Let $z = e^{2\pi iit}$. Then:

$$E = (1, z, z^2, z^3, \ldots),$$

M. Riesz: $f \in H^p$ has basis expansion

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n \qquad (*)$$

Consequence. H^p can be realized as:

A Banach space of functions analytic on the open unit disc !!

?? H^1 ??

Not "complemented" in L^1 (!!)

E NOT a basis for H^1

P. Billard (1971): H^1 has a basis.

P. Wojtaszczyk (1980):

H^1 has an *unconditional* basis.

Two Kinds of Subspaces

Let X be a (closed) subspace of B.

Defn. To say X is *complemented* (in B) means: $B = X \oplus Y$ for some subspace Y.

Prop. *A subspace is complemented iff it is the range of a projection.*

Proof. ($\Leftarrow$)
If X is the range of projection P then

$$B = X \oplus Y \text{ for } Y = (I - P)B.$$

Hardy Spaces again. $1 < p < \infty$

Recall: (a) $e_n(t) = e^{-2\pi int}$

(b) $(e_0, e_1, e_{-1}, e_2, e_{-2}, \ldots)$ is a basis for L^p.

(c) $(e_0, e_1, e_2, \ldots)$ a basis for H^p.

Paley's Theorem. If $1 \le p < 2$, then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^p \implies \sum_{k=1}^{\infty} |a_{2^k}|^2 < \infty$$

Remark. Trivial for $2 \le p$, b/c $L^p \subset L^2$

Corollary. *The H^p-closure of span $(e_{2^n})_0^\infty$ is a complemented subspace isomorphic to Hilbert space.*

"Proof." For $1 < p \le 2$, Paley's Thm. $\implies$ $P : H^p \to H^p$ defined by

$$Pf = \sum_{n=0}^{\infty} a_{2^k} z^{2^k}$$

is a projection with range $= \overline{\text{span}}(e_{2^n})$.

For $2 \le p < \infty$, same true by *duality*.

$$P^* : H^p \to H^p \text{ "="} P.$$

□

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