# The $\overline{\partial}$ Equation

Joel H. Shapiro

Analysis Seminar
Portland State University
February 19, 2021

# Differentiability

**Notation**

$$z = x + iy \in \mathbb{C}$$
$$z = (x, y) \in \mathbb{R}^2$$

$$f = (u, v)\colon \mathbb{R}^2 \to \mathbb{R}^2$$
$$f = u + iv\colon \mathbb{C} \to \mathbb{C}$$

**Definition** [$f$ (real) diff'ble at $z_0 \in \mathbb{R}^2$]: $\exists$ linear transformation $Df(z_0)\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$f(z_0 + h) = f(z_0) + Df(z_0)h + \mathrm{o}(h)$$

as $h \to 0$.

**Definition** [$f$ (complex) diff'ble at $z_0$]
$\exists\ f'(z_0) \in \mathbb{C}$ such that

$$f(z_0 + h) = f(z_0) + f'(z_0)h + \mathrm{o}(h)$$

as $n \to 0$.

**Prop.** *If $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is (real) diff'ble at $z_0 \in \mathbb{R}^2$, then partial derivatives $f_x$ & $f_y$ exist at $z_0$, and for $h \in \mathbb{C}$:*

$$Df(z_0)h = f_x(z_0)\operatorname{Re} h + f_y(z_0)\operatorname{Im} h$$

**Calculation.** At $z_0$:

$$Df = f_x \tfrac{h+\overline{h}}{2} + f_y \tfrac{h-\overline{h}}{2i}$$
$$= \tfrac{1}{2}[f_x - i f_y]h + \tfrac{1}{2}[f_x + i f_y]\overline{h}$$

**Notation:**

$$f_z := \tfrac{1}{2}\left(\tfrac{\partial f}{\partial x} - i\tfrac{\partial f}{\partial y}\right)$$
$$f_{\overline{z}} := \tfrac{1}{2}\left(\tfrac{\partial f}{\partial x} + i\tfrac{\partial f}{\partial y}\right)$$

**Proposition.** *If $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ is (real) diff'ble at $z_0 \in \mathbb{R}^2$, then partial derivatives $f_x$ & $f_y$ exist at $z_0$, and for $h \in \mathbb{C}$:*

$$Df(z_0)h = f_z(z_0)h + f_{\overline{z}}(z_0)\overline{h}$$

# Complex Differentiability

**Notation**

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = f_{\overline{z}} = \frac{\partial f}{\partial \overline{z}} = \overline{\partial} f$$

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = f_z = \frac{\partial f}{\partial z} = \partial f$$

We just proved:

**Proposition.** *If $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ is (real) diff'ble at $z_0 \in \mathbb{R}^2$, then partial derivatives $f_x$ & $f_y$ exist at $z_0$, and for $h \in \mathbb{C}$:*

$$Df(z_0)h = f_z(z_0)h + f_{\overline{z}}(z_0)\overline{h}$$

**Theorem**[Complex Differentiability]

$f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is *complex* diff'ble at $z_0$

$$\Longleftrightarrow$$

$f$ is real diff'ble at $z_0$, and $\overline{\partial} f(z_0) = 0$.

In this case, $f'(z_0) = \frac{\partial f}{\partial z}(z_0) = \partial f(z_0)$

**The Cauchy-Riemann Equation(s)**

$\overline{\partial} f = 0$, i.e.,

$u_x = v_y \;\&\; v_x = -u_y$

**Definition** $f\colon \mathbb{C} \to \mathbb{C}$ *analytic* at $z_0 \in \mathbb{C}$ means:

$f$ is complex differentiable *in a neighborhood of $z_0$*,

i.e., in a neighborhood of $z_0$:

$f$ is real differentiable and $\overline{\partial} f = 0$.

**From now on:** $\Omega$ denotes an open, nonempty subset of $\mathbb{C}$, and $f \in C^1(\Omega)$.

Thus: $f$ *analytic on* $\Omega$ *iff* $\overline{\partial} f = 0$ *at each point of* $\Omega$.

# $\overline{\partial}$ Properties

Suppose $f, g \in C^1(\Omega)$. Then on $\Omega$:

(a) $\overline{\partial}$ *is linear*:

$$\overline{\partial}(af + bg) = a\,\overline{\partial}f + b\,\overline{\partial}g$$

for all scalars $a, b \in \mathbb{C}$.

(b) $\overline{\partial}$ *is a "derivation"*:

$$\overline{\partial}(fg) = f\,\overline{\partial}g + (\overline{\partial}f)g$$

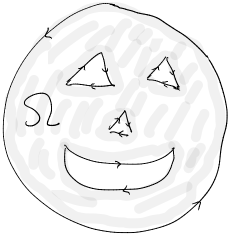
(c) If $f$ is analytic on $\Omega$ then

$$\overline{\partial}(fg) = f\,\overline{\partial}g$$

**Green's Theorem.** *If $f \in C^1(\mathbb{C})$ and $\Omega$ is an open subset of $\mathbb{C}$ with "nice" boundary $\gamma$, then*

$$\int_\gamma f(z)\,dz = 2i\int_\Omega (\overline{\partial}f)\,dA$$

[$dA$ = Lebesgue measure on the plane.]




*Proof.*

$$\int_\gamma f\,dz := \int_\gamma \underbrace{f}_{P}\,dx + \underbrace{(if)}_{Q}\,dy$$

$$= \int_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

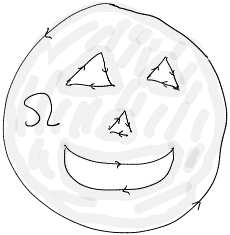
$$= i\int_\Omega \underbrace{\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)}_{2\,\overline{\partial}f} dA$$

# The "Cauchy-Pompieu" formula

**Theorem.** *If* $f : \mathbb{C} \to \mathbb{C}$ *is* $C^1$, *with* $\gamma$ *and* $\Omega$ *as in Green's Theorem, then:*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz \; - \; \frac{1}{\pi} \int_\Omega \frac{\overline{\partial} f(z)}{z - z_0}\, dA$$

*for all* $z_0 \in \Omega$.




**Corollaries.** $\forall\, z_0 \in \Omega$

*Cauchy*: $f$ analytic on $\Omega$

$$\Longrightarrow$$

$$f(z_0) \;=\; \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz$$

*Pompieu*: $f \in C^1(\mathbb{C})$ w/ compact support

$$\Longrightarrow$$

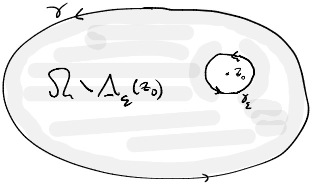
$$f(z_0) \;=\; -\frac{1}{\pi} \int_\Omega \frac{\overline{\partial} f(z)}{z - z_0}\, dA$$

# Proof of the Cauchy-Pompieu formula.

**Theorem.** *If $f\colon \mathbb{C} \to \mathbb{C}$ is $C^1$, with $\gamma$ and $\Omega$ as in Green's Theorem, then:*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz \; - \; \frac{1}{\pi} \int_\Omega \frac{\overline{\partial} f(z)}{z - z_0}\, dA$$

*for all $z_0 \in \Omega$.*




*Proof.* Fix $\varepsilon > 0$ and let $\Delta_\varepsilon = \{|z - z_0| \le \varepsilon\}$, with its boundary $\gamma_\varepsilon$ oriented positive ly.

By *Green's Thm* with $f(z)/(z - z_0)$ for $f(z)$:

$$\int_{\gamma - \gamma_\varepsilon} \frac{f(z)}{z - z_0} dz = 2i \int_{\Omega \setminus \Delta_\varepsilon} \frac{(\overline{\partial} f)(z)}{z - z_0} dA(z)$$

i.e.,

$$\int_{\gamma_\varepsilon} \frac{f(z)}{z - z_0} dz = \int_\gamma \frac{f(z)}{z - z_0} dz - 2i \int_{\Omega \setminus \Delta_\varepsilon} \frac{(\overline{\partial} f)(z)}{z - z_0} dA(z)$$

*Parameterize* $\gamma_\varepsilon$: $z = z_0 + \varepsilon e^{i\theta}$. Then LHS =

$$\int_0^{2\pi} \frac{f(z_0 + \varepsilon e^{i\theta})}{\varepsilon\, e^{i\theta}} i\varepsilon\, e^{i\theta}\, d\theta \to 2\pi i f(z_0) \quad (\varepsilon \to 0+)$$

Last integral on RHS $\to \int_\Omega$ as $\varepsilon \to 0$.
[Reason: $|z - z_0|^{-1}$ is $dA$-integrable over $\Omega$].

*Conclude*:

$$2\pi i f(z_0) = \int_\gamma \frac{f(z)}{z - z_0} dz - 2i \int_\Omega \frac{(\overline{\partial} f)(z)}{z - z_0} dA(z)$$

# The $\overline{\partial}$ Equation

Recall: *Pompieu's Formula*:

*If* $f \in C_c^1(\mathbb{C})$ *then* $\forall\ z \in \mathbb{C}$

$$f(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{(\overline{\partial} f)(\zeta)}{\zeta - z}\, dA(\zeta)$$

$$= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{(\overline{\partial} f)(\zeta + z)}{\zeta}\, dA(\zeta)$$

**The $\overline{\partial}$ Theorem**. Suppose $f \in C^1(\mathbb{C})$. Then: $\exists u \in C^1(\mathbb{C})$ *such that* $\overline{\partial} u = f$ on $\mathbb{C}$.

*Proof* (For $f$ with compact support).

Define $u \in C^1(\mathbb{C})$ by:

$$u(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta + z)}{\zeta}\, dA(\zeta)$$

Apply $\overline{\partial}$ to both sides of this equation & pass it through the integral:

$$(\overline{\partial} u)(z) = -\frac{1}{\pi}\, \overline{\partial}_z \int \frac{f(\zeta + z)}{\zeta} dA(\zeta)$$

$$= -\frac{1}{\pi}\int (\overline{\partial}_z f)(\zeta + z)\frac{dA(\zeta)}{\zeta}$$

$$= -\frac{1}{\pi}\int \frac{(\overline{\partial} f)(\zeta + z)}{\zeta} dA(\zeta)$$

$= f(z)$ as desired.

*Proof* (for $f \notin C^1(\mathbb{C})$): Handwave!!

# Ideals in the Ring of Entire Functions

**Notation.** $\mathscr{E}$ is the *ring of entire functions*.

**Definition.** An *ideal* in a ring is a subring that's closed under multiplication *by elements of the big ring*.

*Consequence*: An ideal is the whole ring iff it contains 1.

**Examples** (in $\mathscr{E}$).

(a) All $f \in \mathscr{E}$ with $f(0) = 0$.

(b) All $f \in \mathscr{E}$ that vanish on a given set.

(c) [A proper ideal with no common zero] The union on $N = 0, 1, 2, \ldots$ of the ideal of functions in $\mathscr{E}$ that vanish on $\{n \in \mathbb{Z} \colon |n| \geq N\}$.

**Question.** In $\mathscr{E}$, does every *finitely generated* ideal have a common zero?

**Theorem** (O. Helmer, 1940) *Every finitely generated proper ideal in $\mathscr{E}$ has a common zero.*

*Proof* (for "doubly generated ideals). Suppose $f_1, f_2 \in \mathscr{E}$ have *no common zero*.

Let $\mathscr{I}$ be the ideal they generate, i.e.,

$$\mathscr{I} = \{f_1 g_1 + f_2 g_2 \colon g_1, g_2 \in \mathscr{E}\}.$$

*To show*: $\mathscr{I} = \mathscr{E}$,

i.e., $\quad 1 \in \mathscr{I}$,

i.e., $\exists g_1, g_2 \in \mathscr{E}$ such that

$$f_1\, g_1 + f_2\, g_2 = 1$$

# Proof of Helmer's Theorem

**Helmer's Theorem.** *Every finitely generated proper ideal in $\mathscr{E}$ has a common zero.*

$\overline{\partial}$-*Proof* (for "doubly generated ideals). Suppose $f_1, f_2 \in \mathscr{E}$ have no common zero.

TO SHOW: $\exists g_1, g_2 \in \mathscr{E}$ such that

$$f_1 g_1 + f_2 g_2 = 1$$

Define:

$$\gamma_j := \frac{\overline{f_j}}{|f_1|^2 + |f_2|^2} \qquad (j = 1, 2)$$

$$\therefore \; \gamma_j \in C^\infty(\mathbb{C}) \quad \& \quad f_1\gamma_1 + f_2\gamma_2 \equiv 1$$

CLAIM: $\exists u \in C^1(\mathbb{C})$ such that

$$g_1 = \gamma_1 + u f_2 \quad \& \quad g_2 = \gamma_2 - u f_1$$

are entire functions (w/ $f_1 g_1 + f_2 g_2 = 1$).

CLAIM. $\exists\; u \in C^\infty(\mathbb{C})$ such that

$$0 = \overline{\partial}(\gamma_1 + u f_2) = \overline{\partial}(\gamma_2 - u f_1)$$

i.e., that

$$\overline{\partial}\gamma_1 + (\overline{\partial}u) f_2 = 0 \qquad (1)$$

$$\overline{\partial}\gamma_2 - (\overline{\partial}u) f_1 = 0 \qquad (2)$$

Multiply (1) by $\gamma_2$ and (2) by $\gamma_1$. Subtract and use $f_1\gamma_1 + f_2\gamma_2 = 1$. Result is:

$$\overline{\partial}u = \gamma_1\,\overline{\partial}\gamma_2 - \gamma_2\,\overline{\partial}\gamma_1 \qquad (3)$$

Solve for $u$ via The $\overline{\partial}$ Theorem (for non-compactly supported RHS !!).

Use (3) & $f_1\,\overline{\partial}\gamma_1 + f_2\,\overline{\partial}\gamma_2 = 0$ to show that $u$ satisfies (1) and (2). $\square$

# Some References


- Matts Andersen, *Topics in Complex Analysis*, Springer 1997.
- Olaf Helmer, *Divisibility properties of integral functions*, Duke Math. J. 6 (1940) 345-356.
- Lars Hörmander, An Introduction to COMPLEX ANALYSIS in Several Variables, van Nostrand 1966.
- Walter Rudin, *Real and Complex Analysis*, 3rd edition; McGraw-Hill 1987.
- Mohamed Jabbari, *Notes for Analysis and Geometry of Several Complex Variables,* Centro de Investigacion en Matematicas (CIMAT), Mexico 2020.
  Available at https://www.cimat.mx/~mohammad.jabbari/course-SCV.pdf