How the Cauchy-Riemann operator makes difficult constructions easy.

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The Cauchy-Riemann Equations (Review)

* Ω an open subset of $\mathbb{C}$

* $f = u + iv \colon \Omega \to \mathbb{C}$ "smooth" on Ω, i.e., partial derivs of u & v exist and are continuous on Ω.

* f is (real) diff'ble on Ω, i.e., $\exists\, A \colon \mathbb{R}^2 \to \mathbb{R}^2$ linear, such that

$$\lim_{h \to 0} \frac{|f(z+h) - f(z) - Ah|}{|h|} = 0$$

* A is unique. We write $A = (Df)(z)$

* f smooth on Ω iff differentiable there with Df continuous there.

* f is (complex) diff'ble on Ω iff

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} := f'(z)$$

exists for each $z \in \Omega$.

Say "f is analytic on Ω."

* $f = u + iv$ (smooth on Ω) is analytic on Ω iff f satisfies the *Cauchy-Riemann equations*

$$u_x = v_y \text{ and } v_x = -u_y \qquad \text{(CR)}$$

on Ω.

Complex Cauchy-Riemann

* $\partial f = \frac{\partial f}{\partial z} := \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})$

  $\overline{\partial} f = \frac{\partial f}{\partial \overline{z}} := \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})$

* (CR) equivalent to $\overline{\partial} f = 0$

  in which case, $f' = \partial f$.

* *Sheldon*: $\overline{\partial}\partial = \frac{1}{4}\Delta$, so:

  u harmonic on Ω

  $\Longrightarrow$

  ∂u analytic on Ω.

* This gives the "right way" to find harmonic conjugates.

* *Logan*: The "Beltrami Equation"

  If $\overline{\partial} u = \mu\, \partial u$, with μ measurable on Ω such that $\|\mu\|_\infty < 1$ a.e., then u is "quasiconformal" on Ω.

* *Joel*: "Nonhomog CR Eqn"

  $$\overline{\partial} u = f \qquad \text{(NHCR)}$$

* Construct an analytic function f with "prescribed behavior" (difficult) by:

  Constructing smooth "prototype" φ with that behavior (easy), and

  "Correcting" φ to $f = \varphi - u$ with u chosen "judiciously" s.t. $\overline{\partial} u = \overline{\partial} \varphi$.

Nonhomogeneous Cauchy-Riemann

Theorem. *$\forall\, f$ smooth on Ω, $\exists\, u$ smooth on Ω with $\overline{\partial} u = f$.*

Last time. Proved for f with compact support. The key was:

The "Fork-in-the-Road Theorem.

Ω a "Green Domain," f smooth on closure of Ω, & $z \in \Omega$ $\implies$

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi}\iint_{\Omega} \frac{\overline{\partial} f(\zeta)}{\zeta - z}\, dA(\zeta)$$

Corollary. (a) f analytic on Ω $\implies$

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

(b) f has compact support $\implies$

$$f(z) = -\frac{1}{\pi}\iint_{\Omega} \frac{\overline{\partial} f}{\zeta - z}\, dA(\zeta)$$

Proof of FTIR Theorem (for spt f compact).
In (b) of Cor.: interchange integral and $\overline{\partial}$.

$$f(z) = \overline{\partial}\underbrace{\left(-\frac{1}{\pi}\iint \frac{f(\zeta)}{\zeta - z}\, dA(\zeta)\right)}_{:=u}$$

For f with non-compact support:

Defer proof (possibly forever)

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Last time: Used NHCR to prove:

Helmer's Theorem (1940).

f_1, f_2 entire, with no common zero
$\implies$
$\exists\, g_1, g_2$ entire with $f_1 g_1 + f_2 g_2 \equiv 1$.

Mittag-Leffler's Theorem

Poles. To say a function g analytic in punctured disc $\Delta = \{0 < |z - z_0| < r\}$ has a *pole* at z_0 of order $n \in \mathbb{N}$" means:

$$\lim_{z \to z_0} (z - z_0)^n f(z) \text{ exists, } \neq 0.$$

i.e., $\exists\, c \in \mathbb{C} \backslash \{0\}$ s.t.

$$f(z) \sim \frac{c}{(z - z_0)^n} \quad \text{as } z \to z_0.$$

SUPPOSE $\{z_k\}_1^\infty$ is a sequence of distinct points in Ω, with no limit point, that $\{n_k\}_1^\infty$ is a sequence of non-negative integers.

Theorem (G. Mittag-Leffler 1876).
$\exists\ f$ analytic on $\Omega \backslash \{z_k\}$ with a pole of order n_k at each z_k.

Gösta Mittag-Leffler (1846-1927)

Proof of Mittag-Leffler's Theorem

For each k choose pairwise disjoint family of open discs $\Delta_k \subset \Omega$ w/ center at z_k, and let Δ'_k be disc w/ same center, half the radius.

Let φ_k be a smooth "bump" function on $\mathbb{C}$ with: $\varphi_k \equiv 1$ on Δ'_k, and $\equiv 0$ off Δ_k.

Graph of φ_k over Δ_k

Our "prototype" function:

$$\varphi := \sum_k \frac{\varphi_k}{(z - z_k)^{n_k}}$$

Note that φ is:

(a) Smooth on $\mathbb{C} \backslash \{z_k\}_1^\infty$,

(b) $\equiv 0$ off $\bigcup_k \Delta_k$, &

(c) Analytic on each $\Delta'_k \backslash \{z_k\}$ with desired pole at z_k.

By (b) and (c): $\overline{\partial}\varphi \equiv 0$ on each Δ'_k, so upon defining Φ to be $\overline{\partial}\varphi$ off the z_k's and 0 on them, φ is smooth Ω.

(!!) $\exists$ smooth u on Ω s.t. $\overline{\partial}u = \Phi$.

"Correct" φ ... by setting $f = \varphi - u$.

On $\Omega \backslash \{z_k\}_1^\infty$: f is smooth & $\overline{\partial}f \equiv 0$.

Thus f is analytic on $\Omega \backslash \{z_k\}_1^\infty$, and ...

f has the "desired pole-behavior."

Consequences of Mittag-Leffler's Theorem

For $\{z_k\}_1^\infty$ and $\{n_k\}_1^\infty$ as in the M-L Thm:

Corollary (Weierstrass) *There is an entire function g with a zero of order n_k at z_k, and no other zero.*

Proof of Corollary. Let $g = 1/f$ with f as in Mittag-Leffler's Theorem. □

NOTATION. $A(\Omega)$ = the ring of functions analytic on Ω.

Helmer's Theorem (1940).

$f_1, f_2 \in A(\Omega)$ *have no common zero*

$$\Longrightarrow$$

$\exists\, g_1, g_2 \in A(\Omega)$ *s.t.* $f_1 g_1 + f_2 g_2 \equiv 1$.

Rephrase conclusion of Helmer's Thm:

The ideal generated by f_1 and f_2 is $A(\Omega)$

NOW SUPPOSE $f_1, f_2 \in A(\Omega)$ have common zero-set Z (w/ multiplicities counted). By Weierstrass $\exists\, f \in A(\Omega)$ with *zero-set* Z.

Corollary. *The ideal generated by f_1 and f_2 is the ideal generated by f.*

Proof. f_1/f & $f_2/f \in A(\Omega)$.

By Helmer $\exists g_1, g_2 \in A(\Omega)$ s.t.

$\frac{f_1}{f} g_1 + \frac{f_2}{f} g_2 = 1, \ i.e., \ f_1 g_1 + f_2 g_2 = f$

i.e., f generates ideal generated by f_1 & f_2.

BY INDUCTION: Every finitely generated ideal of $A(\Omega)$ is "principal".

Some Loose Ends

1. Not every ideal in $A(\Omega)$ is principal.

   Example (for $\Omega = \mathbb{C}$):

   $$\mathscr{I} = \bigcup_N \{f \in A(\mathbb{C})\colon f(n) = 0 \ \forall\, |n| \geq N\}$$

   has no common zero, but is not all of $A(\mathbb{C})$, so not principal.

2. Characterize "non-algebraically" the principal ideals of $A(\Omega)$.

   Is every *maximal* ideal principal?

3. Prove the existence of solutions to $\overline{\partial} u = f$ (already done for f with compact support).

Some References

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