Liouville's Theorem

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The Theorem

Setting: f is an *entire function*, i.e.,

$f'(z_0)$ exists $\forall z_0 \in \mathbb{C}$.

Equivalently:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \forall\ z \in \mathbb{C}.$$

Liouville's Theorem (1847). Bounded entire functions are constant.

Corollary (The Fund'l Thm of Algebra). *Every nonconstant polynomial has a zero.*

Proof of Corollary. Suppose p a nonconstant polynomial.

$$p(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0$$

Then $|p(z)| \sim |z|^d$ as $|z| \to \infty$.

If p has no zero in $\mathbb{C}$, then

$f = 1/p$ is entire, and

$|f| \sim |z|^{-d} \to 0$ as $|z| \to \infty$

Conclude: f is a bounded entire function,

$\therefore f \equiv$ constant.

$\therefore p \equiv$ constant. XX $\square$

Generalized Liouville Theorem

Theorem. *Suppose* $0 \le \rho < \infty$ *and* f *is entire with* $|f(z)| = O(|z|^\rho)$*. Then* f *is a polynomial of degree* $\le \rho$*.*

Proof. Given: entire f and positive numbers A and ρ such that $|f(z)| \le A|z|^\rho$ for each $z \in \mathbb{C}$.

Recall: For $z = re^{i\theta} \in \mathbb{C}$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$$

where the series converges uniformly on each compact subset of $\mathbb{C}$.

Thus for $k = 0, 1, 2, \ldots$

$$\int_0^{2\pi} f(re^{i\theta}) e^{-ik\theta} \frac{d\theta}{2\pi} = \int_0^{2\pi} \sum_{n=0}^{\infty} a_n r^n e^{in\theta} e^{-ik\theta} \frac{d\theta}{2\pi} = \sum_{n=0}^{\infty} a_n r^n \int_0^{2\pi} e^{i(n-k)\theta} \frac{d\theta}{2\pi}$$

$$\therefore \quad a_k r^k = \int_0^{2\pi} f(re^{i\theta}) e^{-ik\theta} \frac{d\theta}{2\pi}$$

$$\therefore \quad |a_k| \le r^{-k} \int_0^{2\pi} \underbrace{|f(re^{i\theta})|}_{\le Ar^\rho} \frac{d\theta}{2\pi} \le \underbrace{Ar^{\rho-k}}_{\to 0 \text{ if } k > \rho}$$

$\therefore \quad |a_k| = 0$ for each $k > \rho$. □

Harmonic Liouville Theorem

Theorem. *Suppose* $u\colon \mathbb{C} \to \mathbb{R}$ *is harmonic, and* $\exists A > 0,\ \rho \geq 0$ *such that*

$$(*) \qquad |u(z)| \leq A\,|z|^\rho \qquad (z \in \mathbb{C}).$$

Then u *is a polynomial of degree* $\leq \rho$.

Proof. $u = \operatorname{Re} f$; $f(z) = \sum_n a_n z^n$ entire.

$$\begin{aligned} u(re^{i\theta}) &= \frac{1}{2}[f(re^{i\theta}) + \overline{f(re^{i\theta})}] \\ &= \sum_{n\in\mathbb{Z}} b_n r^{|n|} e^{in\theta} \end{aligned}$$

where

$$b_n = \begin{cases} a_n/2 & \text{if } n > 0 \\ \overline{a_n}/2 & \text{if } n < 0 \\ \operatorname{Re} a_0 & \text{if } n = 0 \end{cases}$$

As before, but now for $k \in \mathbb{Z}$

$$b_k r^{|k|} = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta}) e^{-ik\theta}\, d\theta$$

$$\therefore \quad |b_k| r^{|k|} \leq \frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\theta})|\, d\theta \leq A r^\rho$$

$\therefore\ |k| > \rho \Rightarrow |b_k| \leq A r^{\rho - |k|} \to 0\ (r \to \infty)$

i.e., $b_k = 0$ whenever $|k| > \rho$.

i.e., u is a polynomial of degree $\leq \rho$. $\square$

Cor. f entire & $u = \operatorname{Re} f$ satisfies (*)
$\implies$ f is a polynomial of degree $\leq \rho$.

Connection with Riemann Zeta Function

GOAL of a previous seminar talk:

> *Show that the Riemann zeta function has infinitely many "nontrivial" zeros.*

Glossary.

(a) An entire function f is "of finite order" if there exists $0 \leq \rho < \infty$ such that $|f(z)| = \mathrm{O}(e^{|z|^\rho})$. The infimum of all such ρ is called the "order" of f.

(b) An entire function of finite order ρ is "of finite type" if there exists $a > 0$ such that $|f(z)| = \mathrm{O}(e^{a|z|^\rho})$.

Proposition. *An entire function of finite order with just finitely many zeros, must be of the form Pe^Q, where P and Q are polynomials.*

I GAVE this "proof" for the Proposition:

- Given: $\exists\ \rho < \infty$ such that $|f(z)| \leq \text{const.}\, e^{|z|^\rho}$.
- Enough to assume: f has *no zeros*.
- $\therefore f = e^Q$ for some entire function Q.
- $\therefore\ \operatorname{Re} Q = \log|f(z)| \leq$ const. $|z|^\rho$
- $\therefore\ \operatorname{Re} Q(z) \leq \text{const.}\, |z|^\rho$
- By generalized harmonic Liouville thm: Q is a polynomial (degree $\leq \rho$).

WHAT'S NOT TO LIKE??

Generalized harmonic Liouville Thm requires $|\operatorname{Re} Q(z)| \leq$ const. $(|z|^\rho)$.

TO SHOW: For $u : \mathbb{C} \to \mathbb{R}$ harmonic on $\mathbb{C}$:

$$u(z) \leq \text{const.}\, |z|^\rho \Rightarrow |u(z)| \leq \text{const.}\, |z|^\rho.$$

To Show: Upper Bound $\Rightarrow$ Lower Bound

Theorem (Borel-Carathéodory). *Suppose u is harmonic on $\{|z| < R\}$ with $u(0) = 0$. If $u(z) \leq M \quad \forall\, |z| < R$ Then :*

$$|u(z)| \leq \frac{2|z|}{R - |z|}\, M \qquad \forall\, |z| < R.$$

Corollary. *Suppose $u\colon \mathbb{C} \to \mathbb{R}$ is harmonic, and $\exists\, \rho \geq 0$ such that $u(z) \leq$ const. $|z|^\rho$ for each $z \in \mathbb{C}$. Then u is a polynomial of degree $\leq \rho$.*

Proof (of Corollary). Fix $z \in \mathbb{C}$.

Use Borel-Carathéodory with

$$R = 2|z| \text{ and } M = \text{const.}\, R^\rho$$

$\therefore \; |u(z)| \leq 2\,\text{const.}\, R^\rho = 2^{\rho+1}\,\text{const.}|z|^\rho$

Now previous argument is OK! $\square$

Example (Illustrates B-C w/ $M = R = 1$) The linear fractional map

$$w = f(z) = 1 - \frac{1-z}{1+z} \; = \; \frac{2z}{1+z}$$

takes $\mathbb{U}$ onto the half-plane $\{\mathrm{Re}\, w < 1\}$, with the unit circle going onto the vertical line $\{\mathrm{Re}\, w = 1\}$.

CONSEQUENTLY:

$$u(z) \; = \; \mathrm{Re}\, f(z) \; = \; \mathrm{Re}\, \frac{2z}{1+z}$$

is harmonic in the unit disc, with

$$u(z) < 1 \qquad (|z| < 1) \quad \text{and}$$

$$|u(z)| = \left|\mathrm{Re}\, \frac{2z}{1+z}\right| \leq \left|\frac{2z}{1+z}\right| \leq \frac{2|z|}{1-|z|}\;.$$

Borel-Carathéodory proof

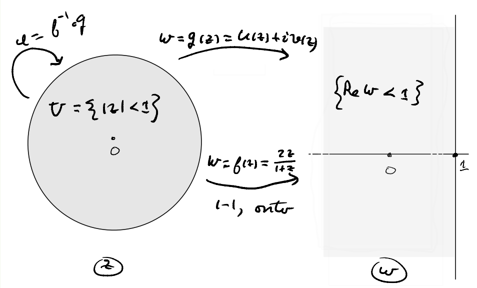
Theorem (Borel-Carathéodory). *If u is harmonic and $< M$ on $\{|z| < R\}$, with $u(0) = 0$, then :*

$$|u(z)| \leq \frac{2|z|}{R - |z|} M \qquad (|z| < R).$$

Proof. WLOG may take $M = R = 1$.

- $u = \operatorname{Re} g$, where g is analytic on $\mathbb{U}$
- $w = g(z)$ maps $\mathbb{U} \to \{\operatorname{Re} w < 1\}$, and
- $g(0) = 0$.

Recall: $w = f(z) = \dfrac{2z}{1+z}$ maps $\mathbb{U}$ *onto* $\{\operatorname{Re} w < 1\}$, and it's *one-to-one*!



Therefore:

- $\varphi := f^{-1} \circ g \colon \mathbb{U} \to \mathbb{U}$,
- $\varphi(0) = 0$, and
- $g = f \circ \varphi \quad$ (!!)

Borel-Carathéodory proof

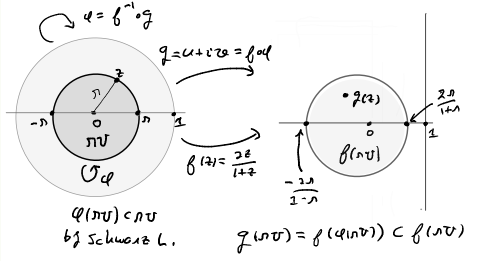
Theorem (Borel-Carathéodory). *If* u *is harmonic and* $< M$ *on* $\{|z| < R\}$, *with* $u(0) = 0$, *then* :

$$|u(z)| \leq \frac{2|z|}{R - |z|} M \qquad (|z| < R).$$

Proof. WLOG may take $M = R = 1$.

- $u = \operatorname{Re} g$, where g is analytic on $\mathbb{U}$, $g(0) = 0$, and g maps $\mathbb{U}$ onto $\{\operatorname{Re} z < 1\}$.
- $w = f(z) = \dfrac{2z}{1+z}$ is a 1-to-1 map taking $\mathbb{U}$ *onto* $\{\operatorname{Re} w < 1\}$.
- $g = f \circ \varphi$ where $\varphi \colon \mathbb{U} \to \mathbb{U}$ and $\varphi(0) = 0$.

The Schwarz Lemma: $\varphi(r\mathbb{U}) \subset r\mathbb{U}$.



$$\therefore \quad \frac{-2r}{1-r} \leq u(z) \leq \frac{2r}{1+r} \qquad (|z| < r)$$

$$\therefore |u(z)| \leq \frac{2|z|}{1-|z|} \qquad (|z| < 1)$$

□

Some Introductions to Complex Analysis

- Donald Sarason, *Complex Function Theory*, American Math. Society 2007
- David Ullrich, *Complex Made Simple*, Amer. Math. Soc. 2008.
- Konrad Knopp, *Theory of Functions Parts I & II*, Dover 1945.
- ———, *Elements of the Theory of Functions*, Dover 1945