Notes on Interpolation of Operators

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These notes concern the "Riesz-Thorin Interpolation Theorem," a special case of which asserts (roughly) that whenever a linear transformation is L^p-bounded for two different values of p, then it is L^p-bounded for every intermediate value of p. In what follows, we'll state this remarkable result in its full generality, and we'll learn how to use it.

1 *Main Theorem*

Notation and terminology

- p and q, possibly with subscripts, are parameters in the extended-real interval $[1, \infty]$. Reciprocals of these parameters will play a crucial role; in this we'll adhere to the convention "$1/\infty = 0$" and "$1/0 = \infty$."
- μ and ν are (non-negative, sigma-finite) measures—possibly on different measure spaces.
- $\mathscr{S}(\mu)$ is the vector space of (μ-equivalence classes of) μ-integrable simple functions. Note that $\mathscr{S}(\mu) \subset L^p(\mu)$ for each $p \geq 1$.
- $\mathscr{M}(\nu)$ is the vector space of (ν-equivalence classes of) ν-measurable functions.
- T is a linear map taking $\mathscr{S}(\mu)$ into $\mathscr{M}(\nu)$.

Definition 1.1. *To say "T is of type (p, q)" means that there is a positive constant $M = M_{p,q}$ such that $\|Tf\|_q \leq M\|f\|_p$ for every $f \in \mathscr{S}(\mu)$.*[1]

[1] Briefly: $T\colon \mathscr{S}(\mu) \to L^q(\nu)$ is "(p, q)-bounded."

Theorem 1.2 (The Riesz-Thorin Interpolation Theorem)**.** *Suppose T is of types (p_0, q_0) and (p_1, q_1). Then:*

M. Riesz [2, 1927], G. O. Thorin [5, 1948]

(a) T is of type (p, q) for each point (p^{-1}, q^{-1}) on the line segment joining (p_0, q_0) to (p_1, q_1), and

(b) More precisely: if there exist positive constants M_0 and M_1 such that for each $f \in \mathscr{S}$

(1) $$\|Tf\|_{q_0} \leq M_0\|f\|_{p_0} \quad \text{and} \quad \|Tf\|_{q_1} \leq M_1\|f\|_{p_1}$$

then for $\theta \in [0, 1]$ such that

(2) $$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

we have

(3) $$\|Tf\|_q \leq M_0^{1-\theta} M_1^{\theta}\, \|f\|_p\,.$$

Remarks 1.3. (a) The Riesz-Thorin Theorem asserts that the *Riesz Diagram* of T—those points (α, β) in the closed unit square of $\mathbb{R}^2$ for which T is of type $(1/\alpha, 1/\beta)$—is convex.

(b) If, in the Riesz-Thorin theorem we denote the indices p and q on the left-hand side of (2) by $p(\theta)$ and $q(\theta)$ respectively, and let $M(\theta)$ denote the resulting $(p(\theta), q(\theta))$-norm of $T \colon \mathscr{S}(\mu) \to L^{q(\theta)}(\nu)$, then the conclusion of our part (b) becomes:

$\log M(\theta)$ *is a convex function on the interval* $0 \le \theta \le 1$.

(c) For part (b) of the Theorem, the special case $\|T\|_{p_0,q_0} = \|T\|_{p_1,q_1} = 1$ implies that for each point (p,q) defined by (2), that $\|T\|_{p,q} \le 1$.

This seemingly special case actually implies the general one! Indeed, if $\|T\|_{p_j,q_j} = M_j$ for $j = 0, 1$, let

$$\tilde{T} := M_0^{t-1} M_1^{-t}\, T\,, \quad \text{whereupon} \quad \|\tilde{T}\|_{p_j,q_j} \le 1 \quad (j = 0, 1),$$

hence $\|\tilde{T}\|_{p,q} \le 1$ from which inequality (3) follows from operator-norm-homogeneity .

2 *Applications: The Identity Map (!)*

Before proceeding to more substantial applications of the Riesz-Thorin Theorem, we'll see how it yields a couple of well-known L^p-norm comparisons.

L^p NORMS: PROBABILITY MEASURES. Suppose μ is a probability measure[2]. Then it's well-known, and (by Hölder's inequality—see Exercise below) not too difficult to prove, that:

[2] i.e., that it has total mass 1.

If $0 < p \le q \le \infty$ then

(4) $$f \in L^q(\mu) \implies f \in L^p(\mu), \quad \text{with} \quad \|f\|_q \le \|f\|_p\,.$$

Proof via Riesz-Thorin Interpolation. The result is obvious if $p = 1$ and q is either 1 or ∞. In other words, the identity map, initially defined on $L^1(\mu)$ is of types $(1,1)$ and $(\infty, 1)$. By the Riesz-Thorin Theorem, the identity map is therefore of type $(q, 1)$ for each $1 \le q \le \infty$, and by the log-convexity inequality (3), its norm, as a mapping from L^q to L^1, is 1. This proves (4) for $p = 1$ and $f \in \mathscr{S}(\mu)$.

For general $f \in L^q$, we know there is a sequence (s_n) chosen from $\mathscr{S}$ such that for a.e. x:

(5) $$s_n(x) \to f(x) \quad \text{and} \quad |s_n(x)| \le |f(x)|\,.$$

Thus for each index n:

(6) $$\|s_n\|_1 \le \|s_n\|_q \le \|f\|_q$$

with the first inequality due to the just-established "type $(q, 1)$-ness" of the identity map, and the second one due to the pointwise a.e. domination of $|s_n|$ by $|f|$. We finish the proof by noting that (with "$\int$" standing in for "$\int \cdot d\mu$")

$$\|f\|_1 = \int |f| = \int \lim_m |s_n| \leq \liminf_n \int |s_n| \leq \|f\|_q$$

with Fatou's Lemma providing the first inequality above, and estimate (6) the second. This establishes (4) for $p = 1$.

This special case leads immediately to the general one. Since the case $p = \infty$ is obvious (due to the probability-ness of the measure μ), we may assume now that $0 < p < \infty$.[3] Fix $f \in L^p(\mu)$ with (for the moment) $\|f\|_p = 1$. Apply, to $g = |f|^p$, the special case with p replaced by 1 and q by q/p. *Conclusion*:

[3] Note, however, that for $p < 1$ the "norm" $\|\cdot\|_p$, while still homogeneous, is no longer subadditive.

$$g \in L^{q/p}(\mu) \quad \text{with} \quad \|g\|_{q/p} \leq 1,$$

i.e., $f \in L^q(\mu)$ with $\|f\|_q \leq 1$.

The desired result for general $f \in L^p(\mu)\backslash\{0\}$, now follows, via "norm-homogeneity", upon applying *this* special case to the function $f/\|f\|_p$. □

Exercise. Use Hölder's inequality to prove (the $p = 1$ case of) statement (4).

Counting measure. Now μ is the counting measure on some at-most countable set. We'll write ℓ^p for $L^p(\mu)$. The result now is the reverse of the "probability-measure" case:

If $0 < p \leq q \leq \infty$ then

$$f \in \ell^p \implies f \in \ell^q \quad \text{and} \quad \|f\|_q \leq \|f\|_p\,.$$

Although this is very easy to prove directly (Exercise), it's worth considering from the Riesz-Thorin perspective. If $p \geq 1$ then the identity map is clearly of type $(1, 1)$ and easily seen to be of type $(1, \infty)$—with norms $= 1$ in both cases. Thus by Riesz-Thorin it's of type $(1, p)$ for each $1 \leq p \leq \infty$, with norm (by log-convexity) ≤ 1. *Exercise*: Use this special case to derive the general one.

3 *Application: The Schur Test*

For the work of this section, fix in your mind a pair of sigma-finite measure spaces, for which we'll denote the respective measures simply by dx and dy (Examples: Lebesgue measure on the real line, counting measure on the an at-most-countable set). We'll study complex-valued "kernels" $K(x,y)$ that are measurable with respect to the sigma-algebras underlying our two measures, and the "integral transformations"

$$T_K f(x) = \int K(x,y) f(y)\, dy \tag{7}$$

defined for those values of x and complex-valued measurable functions f for which the integral makes sense.

Example. Suppose dx and dy are counting measure on the first n positive integers. Then K is an $n \times n$ matrix, and T_K is the linear transformation this matrix induces by left multiplication on (column-vector) $\mathbb{C}^n$.

Theorem 3.1 (The Schur Test). *Suppose there exists positive (finite) constants M_0, M_1 such that*

Issai Schur [4, 1911]

$$\int |K(x,y)|\, dx \leq M_0 \tag{8}$$

for dy-a.e. y, and

$$\int |K(x,y)|\, dy \leq M_1 \tag{9}$$

for dx-a.e. x. Then for each $p \in [1,\infty]$, the operator T_K defined by (7) *takes $L^p(dx)$ boundedly into $L^p(dy)$, with*

$$\|T_k f\|_p \leq M_0^{1-1/p} M_1^{1/p}$$

for each $f \in L^p(dx)$ (i.e., $\|T_K\|_{p,p} \leq M_0^{1/p'} M_1^{1/p}$).

Proof. From inequality (8) and Fubini's Theorem we see that for each $f \in L^1(dx)$:

$$\begin{aligned} \int \left(\int |K(x,y)|\, |f(y)|\, dy \right) dx &= \int \left(\int |K(x,y)|\, dx \right) |f(y)|\, dy \\ &\leq M_0 \int |f(y)|\, dy = M_0 \|f\|_1 , \end{aligned}$$

hence T_K maps $L^1(dy)$ boundedly into $L^1(dx)$, with $\|T_K\|_{1,1} \leq M_0$. From inequality (9) we have for each $f \in L^\infty(dy)$ and dx-a.e. x:

$$|T_K f(x)| \leq \int |K(x,y)|\, |f(y)\, dy \leq M_1 \|f\|_\infty .$$

Thus T_K maps $L^\infty(dy)$ boundedly into $L^\infty(dx)$, with norm $\|T_K\|_{\infty,\infty} \leq M_1$.

Conclusion: T_K is of types $(1,1)$ and (∞,∞) so by the "qualitative" Riesz-Thorin Theorem it's of type (p,p) for each $p \in [1,\infty]$. This, along with the "quantitative" amendment to Riesz-Thorin, shows that:

$$\|T_K f\|_p \leq M_0^{1-1/p}\, M_1^{1/p}\, \|f\|_p , \tag{10}$$

at least for each $f \in \mathscr{S}$. The result now follows for $f \in L^p$ upon applying (10) to a sequence of integrable simple functions that "adaptively approximates" f, followed by an adroit application of Fatou's Lemma. □

The special case $p = 2$ of the Schur test is particularly easy to state:

Corollary 3.2. *Suppose the "integral kernel K satisfies the hypotheses of the Schur test. Then T_k maps $L^2(dx)$ boundedly into $L^2(dy)$, with operator norm $\le \sqrt{M_0\, M_1}$.*

What does this say about the operator norm of an $m \times n$ matrix as it maps one euclidean space into another?

Convolution operators. Let's now confine our attention to the spaces L^p defined for Lebesgue measure on the real line $\mathbb{R}$—or, if you prefer, on euclidean space $\mathbb{R}^n$.[4]

[4] Or, if you prefer, on any locally compact abelian group with Haar measure.

Definition 3.3 (Convolution)**.** *The* convolution *of f and g in L^1 at $x \in \mathbb{R}$, is:*

$$(f * g)(x) := \int f(x-t)g(t)\,dt\,. \tag{11}$$

For this definition to make sense we need to show that the integrand on the right-hand side of (3.3) is integrable for a.e. x. Here's why:

Lemma 3.4. *If f and g belong to L^1, then:*

(a) The integrand on the right-hand side of (11) *is integrable for a.e. $x \in \mathbb{R}$ (so the left-hand side is defined for a.e. x).*

(b) $(f * g) \in L^1$, and

(c) $\|f * g\|_1 \le \|f\|_1\, \|g\|_1$.

Proof. We may without loss of generality assume that both f and g are everywhere-defined, and take only non-negative values. Thus $(f * g)(x)$ is defined for each x in $[0, 1]$, but possibly its value is infinite! Nevertheless, Fubini's Theorem applies, and yields

$$\int \left(\int f(x-t)g(t)\,dt \right) dx = \int \left(\int f(x-t)\,dx \right) g(t)\,dt\,.$$

Thanks to the translation-invariance of Lebesgue measure, the inner integral on the right is just $\int f$, which proves everything that was promised. □

Definition 3.5 (Convolution operator)**.** *For $f \in L^1$ define $C_f \colon L^1 \to L^1$ by*

$$C_f\, g = f * g \qquad (g \in L^1).$$

Lemma 3.4 justifies our contention that the *convolution operator* maps L^1 into itself, and shows that its norm is $\le \|f\|_1$.[5] Even more is true:

[5] In fact, its norm *is* $\|f\|_1$, but we won't prove this here.

Corollary 3.6. *If $f \in L^1$ and $1 \le p \le \infty$, then C_f maps L^p boundedly into itself, with norm $\le \|f\|_1$.*

In other words:

If $f \in L^1$ and $g \in L^p$, then $f * g \in L^p$ with $\|f * g\|_p \le \|f\|_1 \|g\|_p$.

Proof. Let

$$K(x,y) = f(x-y) \qquad (x, y \in \mathbb{R})$$

and note that our "integral kernel" K satisfies the hypotheses of the Schur Test with $M_0 = M_1 = \|f\|_1$. □

Young's Inequality. A simple change-of-variable shows, thanks to the translation-invariance of Lebesgue measure, that *convolution on L^1 is commutative*, i.e., that

$$f * g = g * f \qquad (f, g \in L^1).$$

Thus, if $1 \le r \le \infty$ and $g \in L^r$, then Theorem 3.6 allows us to think of C_g as mapping L^1 boundedly into L^r, with operator-norm $\le \|g\|_r$. An application of Hölder's Inequality shows that C_g maps $L^{r'}$ boundedly into L^∞, with operator-norm $\le \|g\|_r$.[6] Thus C_g is of types $(1, r)$ and (r', ∞), i.e., the "reciprocal points" $(1, 1/r)$ and $(1/r', 0)$ belong to the Riesz Diagram[7] of C_g, hence by Riesz-Thorin, so does every point (x, y) on the line segment joining these points. Since this line segment has equation

$$y = x - \frac{1}{r'} = x + \frac{1}{r} - 1 \qquad \Big(\frac{1}{r'} \le x \le 1\Big),$$

we've shown that given p, q with $1 \le p \le q \le \infty$, and $g \in L^r$ with

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{r'}$$

then C_g is of type (p, q). This, along with the log-convexity of norms, and the end-game argument involving adapted sequences of simple functions and Fatou's Lemma, proves:

Theorem 3.7 (Young's Inequality). *If $1 \le p \le q \le \infty$ and $g \in L^r$ with*

$$\frac{1}{r'} = \frac{1}{p} - \frac{1}{q}$$

then C_g maps L^p boundedly into L^q, with operator norm $\le \|g\|_r$.

W. H. Young [6, 1912)]

...i.e., if $f \in L^p$ and $g \in L^r$, then $f * g \in L^q$ with $\|f * g\|_q \le \|f\|_p \|g\|_r$

[6] Here r' is the Hölder conjugate of r, i.e., $1/r + 1/r' = 1$

[7] See Remark 1.3(a).

4 *Applications: Fourier Series*

Our object of interest is the *Fourier transform* of a function $f \in L^1$, defined for $n \in \mathbb{Z}$ by

$$\hat{f}(n) = \int f(t)e^{-2\pi int}\,dt\,,$$

where $\int$ stands for $\int_0^1$.

THE RIESZ TRANSFORM Last term we studied the *Riesz transform* R, defined initially on L^2 by

$$\widehat{RF}(n) = \frac{1}{i}\sigma(n)\hat{f}(n) \qquad (f \in L^2, n \in \mathbb{Z}),$$

where σ is the *step function* on $\mathbb{Z}$ that takes value -1 on each negative integer, $+1$ on each positive integer, and 0 at 0.[8] By Parseval's Theorem, the Riesz transform is a bounded linear contraction on L^2.

We saw last term how Marcel Riesz used some elementary algebraic trickery to show that R is a bounded operator from L^p to L^p whenever p is an even positive integer, after which the Riesz-Thorin Theorem gave "for free" the same result for each real $p \in [2, \infty)$ [3,1928]. The same result for $1 < p < 2$ then followed from a duality argument based on the fact that for $2 < p < \infty$, the adjoint $R^*\colon L^{p'} \to L^{p'}$ of $R\colon L^p \to L^p$ is again R.

We then saw how this boundedness for the Riesz transform implied that for $1 < p < \infty$, each function in L^p is the L^p-norm limit of its Fourier series.

[8] *Motivation*: For $f \in L^2$, say, with Fourier series $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\theta}$, its Riesz Transform has Fourier series $\tilde{f} = \frac{1}{i}\sum_{n\in\mathbb{Z}} \sigma_n \hat{f}(n)e^{in\theta}$, hence $\frac{1}{2}(f + i\tilde{f})$ has Fourier series $\sum_{n\geq 0} \hat{f}(n)e^{in\theta}$.

THE HAUSDORFF-YOUNG INEQUALITIES. Let $\mathscr{F}$ denote the map $f \to \hat{f}$ that associates each $f \in L^1$ to its Fourier transform $\hat{f}\colon \mathbb{Z} \to \mathbb{C}$. We call $\mathscr{F}$ the *Fourier Transform*: it's clearly a linear contraction taking L^1 into the sequence space $\ell^\infty(\mathbb{Z})$, and by Parseval's identity, it's a linear isometry $L^2 \to \ell^2(\mathbb{Z})$.

Now $\ell^p(\mathbb{Z})$ is just $L^p(\nu)$, where ν is counting measure on the integers, so with this setup the Fourier transform is of types $(1, \infty)$ and $(2, 2)$. By the Riesz-Thorin Theorem, it's of type (p, q) where the point $(1/p, 1/q)$ lies on the line segment joining $(1/2, 1/2)$ to $(1, 0)$, i.e., the line segment $x + y = 1$, $1/2 \leq x \leq 1$.

CONCLUSION: *For* $1 \leq p \leq 2$ *and* $f \in \mathscr{S}$:

$$\|\hat{f}\|_q \leq \|f\|_p$$

where q *is the Hölder-conjugate index for* p.

To extend this inequality to a fixed $f \in L^p$ we need only choose a sequence (s_n) of integrable simple functions adapted to f, use

Fatou's Lemma as before, and observe that the convergence of s_n to f in L^p implies pointwise convergence of $\hat{s_n}$ to $\hat{f}$ (Exercise).

Upon stripping away the Banach-space setting we obtain

Theorem 4.1 (Hausdorff-Young). *Suppose* $f \in L^p$ *where* $1 \le p \le 2$. *Then* $\hat{f} \in \ell^q(\mathbb{Z})$ *where* $\frac{1}{p} + \frac{1}{q} = 1$. *Moreover:*

W. H. Young [7, 1913], F. Hausdorff [1, 1923]

$$\left(\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^q \right)^{1/q} \le \left(\int |f|^p \right)^{1/p}$$

Exercise. Show that for $2 \le p \le \infty$ the Hausdorff-Young inequalities "reverse", i.e., that whenever $(a_n) \in \ell^p(\mathbb{Z})$ there exists $f \in L^q$ with $\hat{f}(n) = a_n$ for each $n \in \mathbb{Z}$, and

$$\left(\int |f|^p \right)^{1/p} \le \left(\sum_{n\in\mathbb{Z}} |a_n|^q \right)^{1/q}$$

Suggestion. There are two possibilities: (a) Use the original result and derive this one by duality.

(b) Observe that the result is easily seen to hold for $p = 1$, and we already know it's true for $p = 2$. Put this in a form to which Riesz-Thorin applies.

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