Conjugate Functions á la Riesz

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ABSTRACT. Sometime during the mid-1920's the famous Hungarian mathematician Marcel Riesz conceived a beautiful collection of theorems. He could prove them all if he could only prove a very special and innocent-looking case of one of them:

Given a complex polynomial $F(z) = u(z) + iv(z)$ with $F(0) = 0$, is there a positive constant C such that

$$(R_4) \qquad \int_0^{2\pi} v(e^{i\theta})^4 d\theta \le C \int_0^{2\pi} u(e^{i\theta})^4 d\theta\,?$$

While an setting an exam question involving integrals like $\int |F|^4$, Riesz wondered what would happen if he removed the absolute values from the integrand. And *voila*; there was his proof, staring him in the face!

What was this proof? Why was Riesz interested? We will investigate.

Anecdote paraphrased from [2], pp. 193–194.

If $v(0)$ were left unconstrained then (R_4) could not hold (example $F \equiv$ an arbitrary complex constant). We're also requiring that $u(0) = 0$. This just a convenience that simplifies arguments; it isn't necessary.

1 *Riesz's proof of* (R_4)

FOR A FUNCTION f integrable over the interval $[0, 2\pi]$ we'll write

$$(1) \qquad \int f = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta\,.$$

which we'll interpret as the integral of f either: "over $[-\pi, \pi]$ with respect to normalized Lebesgue measure," or "over the unit circle $\mathbb{T}$ with respect to normalized arclength measure." Note in particular that if

$$(2) \qquad e_n(\theta) := e^{in\theta} \qquad (m \in \mathbb{Z},\ \theta \in \mathbb{R}),$$

then

$$(3) \qquad \int e_n = \begin{cases} 0 & \text{if } n \ne 0 \\ 1 & \text{if } n = 0\,. \end{cases}$$

Given a general complex polynomial $F(z) = \sum_{k=0}^{n} c_n z^n$, we'll let $u(z)$ denote its real part and $v(z)$ its imaginary part. Thus $F = u + iv$ with u and v being real-valued harmonic functions linked by the Cauchy-Riemann equations. From the work above on complex exponentials, we see that

Henceforth we'll use the algebraist's notation $F \in \mathbb{C}[z]$ for such polynomials.

$$(4) \qquad \int F = c_0 = F(0)\,.$$

PROOF OF (R_4). Since F is a polynomial with $F(0) = 0$, the same is true of F^4, hence by (4):

$$\int F^4 = 0\,.$$

Consequently

$$0 = \text{Re} \int F^4 = \int \text{Re}\,(u+iv)^4 = \int (u^4 - 6u^2v^2 + v^4)\,,$$

so that

$$\int v^4 = -\int u^4 + 6\int u^2v^2 \tag{5}$$

Equation (5) tells us that if u were $\equiv 0$ then so would be v, in which case (R_4) would follow trivially. Thus we may assume that u is not the zero-polynomial, and consequently that $\int u^4 > 0$.

On the right-hand side of (5) let's erase the minus sign on the first integral, and apply the Cauchy-Schwartz inequality to the second one. There results:

$$\int v^4 \le \int u^4 + 6\left(\int u^4\right)^{1/2}\left(\int v^4\right)^{1/2}\,. \tag{6}$$

To finish the proof, let X be the quotient of $\int v^4$ by $\int u^4$ (the latter integral, as we've already observed, being > 0). i.e., let

$$\int v^4 = X\int u^4\,.$$

Upon substituting this expression into inequality (6), and cancelling the positive quantity $\int u^4$ from both sides, we obtain

$$X \le 1 + 6\,X^{1/2}\,,$$

hence $X \le (3+\sqrt{10})^2 < 38$.

IN SUMMARY, we've proved—as did Riesz:

If $F = u + iv \in \mathbb{C}[z]$ with $F(0) = 0$, then

$$\int v^4 \le 38\int u^4,$$

hence (R_4) is true with $C_4 = 38$.[1] □

[1] The best-possible constant (assuming only $v(0) = 0$) was found in the early 1970's by Brian Cole (unpublished) and Stylianos Pichorides [4]; it is $\cot(\pi/8)^4 = 33.97056\cdots$.

Exercise. Show that Riesz's argument works, even more simply, to prove that for each complex polynomial $F = u + iv$ with $F(0) = 0$:

$$\int v^2 \le \int u^2\,. \tag{R_2}$$

2 Why (R_4)?—Part I

One can think of inequalities like (R_4) and its generalizations (R_p) to follow as providing bounds for solutions to the *Cauchy-Riemann* system of partial differential equations

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

where u is, in our case, a harmonic polynomial in two real variables, and v is its *normalized harmonic conjugate* (i.e., the unique harmonic function v (in this case also a polynomial) such that $v(0) = 0$ such that $F = u + iv$ is analytic (in our case: a polynomial in $z = x + iy$).

Why should it be important to obtain such bounds on a single circle? As happens frequently, a seemingly special case gives something much more general. To see how this happens for Riesz's problem, fix $F = u + iv = \sum_{n=1}^{N} a_n z^n$ a polynomial with $F(0) = 0$, and for $r \geq 0$ let F_r be the polynomial defined by

$$F_r(z) = F(rz) = \sum_{n=1}^{N} a_n r^n z^n$$

The decomposition of F_r into real and imaginary parts is then

$$F_r = u_r + iv_r$$

with $u_r(z) = u(rz)$ and similarly for v_r. Thus inequality (R_4) applies as well to the pair u_r, v_r , and yields

$$\int_{-\pi}^{\pi} |v(re^{i\theta})|^4 \, d\theta \leq C_4 \int_{-\pi}^{\pi} |u(re^{i\theta})|^4 \, d\theta \qquad (r \geq 0),$$

so the L^4-mean of v is not just controlled by that of u on one circle, it's controlled by u on *every* circle centered at the origin.

We can go further by integrating both sides of the above inequality on the measure rdr over, say, the unit interval. The result then is

$$\int_{\mathbb{U}} |v|^4 \, dA \leq C_4 \int_{\mathbb{U}} |u|^4 \, dA \,,$$

where $\mathbb{U}$ is the open unit disc $\{|z| < 1\}$ (with a similar result for any disc centered at the origin).

Exercise. Show that $\int |v|^4 \, d\mu \leq C_4 \int |u|^4 \, d\mu$ for any "radially symmetric" positive measure μ on the plane.

3 Beyond (R_4)—Part I

WHAT WAS the "beautiful collection of theorems" that Riesz knew would follow from (R_4)? Here's a beginning:

Proposition 3.1. *Suppose $F = u + iv \in \mathbb{C}[z]$ with $v(0) = 0$. Then for each positive integer n there exists a constant $C = C_{2n}$, independent of F, such that*

$$\int v^{2n} \leq C \int u^{2n}. \qquad (R_{2n})$$

We've just established this result for $p = 4$ as an Exercise, and have noted in an Exercise that the case $p = 2$ is even simpler. The case $p = 2n$ follows along the same lines; for simplicity we'll just give the argument for $p = 6$, leaving the general case as an exercise.

PROOF OF (R_6). Proceeding as in the the case $p = 4$ we observe that

$$0 = F(0)^6 = \int (u + iv)^6$$

so the integral on the right is real. From this observation and the Binomial Theorem we see that

$$0 = \int \mathrm{Re}\,(u + iv)^6 = \int (u^6 - 15u^4v^2 + 15u^2v^4 - v^6)$$

whereupon

$$\begin{aligned} \int v^6 &= -\int u^6 + 15\int u^4v^2 - 15\int u^2v^4 \\ &\leq \int u^6 + 15\int u^4v^2 + 15\int u^2v^4. \end{aligned}$$

From Hölder's inequality we find

$$\int u^4v^2 \leq \left(\int u^6\right)^{2/3}\left(\int v^6\right)^{1/3} \text{ and } \int u^2v^4 \leq \left(\int v^6\right)^{2/3}\left(\int u^6\right)^{1/3}.$$

This, along with the previous inequality yields

$$\int v^6 \leq \int u^6 + 15\left(\int u^6\right)^{2/3}\left(\int v^6\right)^{1/3} + 15\left(\int v^6\right)^{2/3}\left(\int u^6\right)^{1/3}$$

Write $\int v^6 = X\int u^6$ and substitute this into the above inequality. We may divide both sides of the resulting inequality by $\int u^6$, since that quantity is > 0.[2] The result is:

$$X \leq 1 + 15X^{1/3} + 15X^{2/3},$$

which can hold only for X in a bounded interval of the non-negative real axis, say $0 \leq X \leq C$. This proves R_6.[3] □

[2] Recall: we are assuming that u is not identically 0.

[3] …with $C \leq 40524$. By [4], the best constant is $C_6 = \cot(\pi/12)^6 \approx 2702$

4 *Beyond* (R_4)*—Part II: The Riesz Transform*

HAVING PROVED the inequalities (R_{2n}) for each positive integer n, could Riesz extend his results to inequalities (R_p) for each $2 < p < \infty$? The answer is "Yes"; we'll explore his remarkable argument in this section.

For $1 \le p < \infty$, recall the L^p *norm* of $f\colon [0, 2\pi] \to \mathbb{C}$ (Lebesgue-measurable), defined by:

$$\|f\|_p := \left(\int |f|^p\right)^{1/p}$$

Integral notation as in (1) of Section 1. Since $|f|^p$ is measurable, its integral makes sense, possibly $= \infty$.

With this notation, inequality (R_{2n}) of Proposition 3.1 can be restated:

For each positive integer n there exists a positive constant C_{2n} such that: If $F = u + iv \in \mathbb{C}[z]$ with $v(0) = 0$, then

$$(R'_{2n}) \qquad \|v\|_{2n} \le C_{2n}\|u\|_{2n}\,.$$

Note that the constant "C" is defined differently here than in the last section.

We know from real analysis that for $1 \le p < \infty$, the space $L^p = L^p([0, 2\pi])$, consisting of those (a.e. equivalence classes of) measurable, complex-valued functions f on $[0, 2\pi]$ for which $\int |f|^p < \infty$, is a Banach space in the norm $\|\cdot\|_p$.

In this setting Riesz's results assert—so far—that if n is a positive integer, then for polynomials $F = u + iv$ with $v(0) = 0$, the linear map $u \to v$ is continuous in the L^{2n} metric. In the language of undergraduate-level complex analysis, we're talking about the map that acts on the (real) vector space of harmonic polynomials that vanish at the origin by taking a vector u to its harmonic conjugate v.

TO GIVE a less clumsy interpretation, let's fix one of our polynomials $F = u + iv$, with $F(0) = 0$, and write it out:

$$F(z) = \sum_{n=1}^{N} a_n z^n = u(z) + iv(z) \qquad (z \in \mathbb{C}).$$

Note that the integrals in inequalities (R_{2n}) extend over very restricted values of z; those in the *unit circle* $z = e^{i\theta}$. Thus, we care about the "trigonometric polynomial"

$$F(e^{i\theta}) = \sum_{n=1}^{N} a_n e^{in\theta} \qquad (\theta \in \mathbb{R}),$$

for which we have

$$(7) \qquad u(e^{i\theta}) = \frac{1}{2}[F(e^{i\theta}) + \overline{F(e^{i\theta})}] = \sum_{n=-N}^{N} b_n e^{in\theta}$$

with

$$b_n = \begin{cases} a_n/2 & \text{if } n \geq 0 \\ 0 & \text{if } n = 0 \\ \overline{a_n}/2 & \text{if } n < 0\,. \end{cases} \tag{8}$$

Similarly

$$v(e^{i\theta}) = \frac{1}{2i}[F(e^{i\theta}) - \overline{F(e^{i\theta})}] = \sum_{n=-N}^{N} \omega_n b_n e^{in\theta} \tag{9}$$

where

$$\omega_n = \begin{cases} 1/i & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -1/i & \text{if } n < 0\,. \end{cases} \tag{10}$$

THE RIESZ INEQUALITIES, as we've just noted, really concern *trigonometric polynomials*. With this in mind, let's "reverse-engineer" our previous setup, beginning not with the polynomial $F = u + iv \in \mathbb{C}[z]$, but rather with u, regarded as a trigonometric polynomial—in fact an *arbitrary* one

$$u(e^{i\theta}) = \sum_{n=-N}^{N} b_n e^{in\theta}\,, \tag{11}$$

for which the coefficients b_n are any complex numbers.

In this generality we define the "conjugate" of u" to be the trigonometric polynomial v defined by:

$$v(e^{i\theta}) = \sum_{n=-N}^{N} \omega_n b_n e^{in\theta}\,, \tag{12}$$

Note that this definition makes the constant coefficient of v equal to 0.

with ω_n defined by (10) above. Thus $F = u + iv$ is, as before, a trigonometric polynomial that is "analytic" in the sense that it's built from only positive powers of $e^{in\theta}$:

But now u and v need not be the real and imaginary parts, respectively of F. How might they differ from these real and imaginary parts?

$$F(e^{i\theta}) = b_0 + 2\sum_{n=1} b_n e^{in\theta}\,.$$

Let's call v the "Riesz transform" of u write $v = Ru$. The definition (12) tells us that R is a (complex-) linear transformation on the (complex) vector space $\mathscr{T}$ of trigonometric polynomials.

A more common notation for v is $\tilde{u}$, in which case it's called the *conjugate function* (and sometimes the *Hilbert transform*) of u.

Proposition 4.1. *For each positive integer n the Riesz transform is continuous in the L^{2n}-metric on $\mathscr{T}$.*

Proof. The inequalities R'_{2n} establish this for the space of *real-valued* trigonometric polynomials with "constant term" zero.[4] The trick is to remove these extra hypothesis.

[4] *Exercise.* Show that R transforms this space into itself.

To remove the "real-ness" hypothesis, fix a trigonometric polynomial u with constant term 0, and write it in real and imaginary parts as $u = u_1 + iu_2$. By complex linearity, $Ru = Ru_1 + iRu_2$. Fix a positive integer n and set $p = 2n$. Thanks to Proposition 3.1 we know (since u_1 and u_2 are both real-valued with constant term $= 0$) that that there is a positive constant C_p such that

$$\|Ru_j\|_p \leq C_p \|u_j\|_p \qquad (j = 1, 2),$$

whereupon

$$\begin{aligned}\|Ru\|_p &= \|Ru_1 + Ru_2\|_p \\ &\leq \|Ru_1\|_p + \|Ru_2\|_p \\ &\leq C_p \left[\|u_1\|_p + \|u_2\|_p\right].\end{aligned}$$

Now upon recalling that $u = u_1 + iu_2$, we have $\|u_j\|_p \leq \|u\|_p$ for $j = 1, 2$. Thus $\|Ru\|_p \leq 2C_p\|u\|_p$, so the complex extension of our previous "real" results works, at the cost of doubling the previously obtained constants.

To remove the "constant term $= 0$" hypothesis, let $u_0 = u - u(0)$, so

$$Ru = R(u_0 + u(0)) = Ru_0 + \underbrace{R(u(0))}_{=0} = Ru_0$$

whereupon

$$\|Ru\|_p = \|Ru_0\|_p \leq C_p\|u_0\|_p = C_p\|u - u(0)\|_p \leq (C_p + 1)\|u\|_p\,,$$

where the last inequality comes from $\|u(0)\|_p \leq \|u\|_p$ (exercise). ☐

Trigonometric polynomials are important for our purposes because *for each $p < \infty$ they are dense in L^p.* [5] Inequalities (R_{2n}) imply that, for each positive integer n, our Riesz operator R is *uniformly continuous* in the norm of L^{2n}. Thus it has a unique extension to a continuous linear transformation on L^{2n}, which we'll continue to denote by R.

[5] See, e.g. the lecture notes [8], freely available online. The proof there is given for L^1; it's (almost) the same for the other L^p spaces.

. . . and about which we'll say more later.

This brings us to a result whose proof Littlewood has called: "The most impudent method in mathematics".[6] Here is the special case that applies to our situation:

[6] See [2], page 40 (last sentence).

Theorem 4.2 (Riesz-Thorin Interpolation—*very* special case)**.** [7] *Suppose $1 \leq p < q < \infty$, and T is a linear transformation defined on L^p (and therefore on L^q), and continuous on both spaces. Suppose $r \in (p, q)$. Then T maps L^r continuously into itself.*

[7] See [9] for the general case, and more generally for an excellent introduction to this notion of "interpolation". In [1], Ch., IV, pp.93–97 you'll find an even more general result, from which the Riesz-Thorin Theorem emerges as a corollary.

The Riesz-Thorin Theorem, allows us to extend Proposition 4.1 to all $p \in [2, \infty)$.

Theorem 4.3. *For $2 \leq p < \infty$ the Riesz transform extends uniquely to a linear transformation continuous on L^p.*

5 *The Riesz transform on L^p: What is it?*

LET'S REVIEW. Given a trigonometric polynomial

$$f(e^{i\theta}) = \sum_{|n| \leq N} a_n e^{in\theta},$$

its *Riesz transform* (or *conjugate function*, or *Hilbert transform*) is the trigonometric polynomial

$$Rf(e^{i\theta}) = \sum_{|n| \leq N} \omega_n \, a_n \, e^{in\theta},$$

with the "coefficient multiplier" ω_n defined by (10) on page 6. Thanks to our work on Riesz's inequalities for p an even integer, we were able to prove that R is L^p-norm-continuous on the vector space of trigonometric polynomials and so—thanks to the density of trigonometric polynomials in L^p—has a unique extension to a continuous linear mapping taking L^p into itself (for p an even integer). The Riesz-Thorin interpolation theorem then allowed us to extend this result to all values $2 \leq p < \infty$.

QUESTIONS REMAIN, for example:

- *What does R look like on the vectors in L^p that are* not *trigonometric polynomials.*
- *What happens if $1 \leq p < 2$?*
- *What happens if $p = 1$ or $p = \infty$?*

We'll answer the first of these questions right now, the second one in the next section, and the third in §8.

NOTATION:

$$e_n(\theta) = e^{in\theta} \qquad (n \in \mathbb{Z}, \; \theta \in \mathbb{R}),$$

and observe that

$$\int e_n = \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} \, d\theta = \begin{cases} 0 & \text{if } \; n \neq 0 \\ 1 & \text{if } \; n = 0 \, . \end{cases} \tag{13}$$

Furthermore, let's impose a natural *inner product* on the vector space $\mathscr{T}$ of trigonometric polynomials

$$\langle f, g \rangle = \int f\,\overline{g} \qquad (f, g \in \mathscr{T}).$$

With this inner product we see from (13) that the exponentials $\{e_n \colon n \in \mathbb{Z}\}$ form an orthonormal basis for $\mathscr{T}$. This exhibits each trigonometric polynomial f as a sum

$$f = \sum_{|n| \le N} c_n e_n$$

with

$$c_n = \langle f, e_n \rangle = \int f\,\overline{e_n} = \int f\,e_{-n}\,.$$

This suggests that for $f \in L^1$ and $n \in \mathbb{Z}$ we define

$$\widehat{f}(n) := \langle f, e_n \rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\,d\theta,$$

Recall that, by Hölder's inequality $L^p \subset L^1$ for each $p > 1$, with $\|\cdot\|_p \le \|_{\|}1\cdot$ on L^p. In particular, the definition here applies in L^p for each $p > 1$.

to be known henceforth as the *n-th Fourier coefficient* of f, and the formal series[8]

$$\sum_{n \in \mathbb{Z}} \widehat{f}(n)e^{in\theta}$$

is called the *Fourier series* of f. Clearly $|\widehat{f}(n)| \le \|f\|_1$ for each $n \in \mathbb{Z}$. The function $\widehat{f} \colon \mathbb{Z} \to \ell^\infty$ is called the *Fourier transform* of f.[9].

[8] The term "formal" refers to the fact that—right now—we're making no predictions about convergence.

[9] *Exercise*. In addition, $|\widehat{f}(n)| \to 0$ as $|n| \to \infty$ (this is the *Riemann-Lebesgue Lemma*).

L^2 is the place to be!

Since L^2 is a Hilbert space in which the trigonometric polynomials are dense, the complex exponentials $\{e_n \colon n \in \mathbb{Z}\}$ form an orthonormal *basis*, meaning that each $f \in L^2$ is uniquely represented by its Fourier series, and that this series converges to f in the metric of L^2. Furthermore we have *Parseval's identity*

$$\|f\|_2^2 = \sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^2$$

for each $f \in L^2$, and indeed, a function $f \in L^1$ lies in L^2 if and only if the sum on the right is finite! [10]

[10] In particular, the convergence of the Fourier series of $f \in L^2$ to f is independent of the way in which the terms of the series are arranged.

What about the other L^p spaces?

It turns out that the Fourier series "representation" is unique for L^1, in that if $\widehat{f}(n) = 0$ for every $f \in L^1$, then f is the zero-vector of L^1 (i.e., $f(e^{i\theta}) = 0$ for almost-every θ). However the Fourier series of an $f \in L^1$ need not converge to f (or to anything) in the L^1-metric.[11]

[11] See §8 below.

The same is true for L^∞. Indeed, if the Fourier Series of a function $f \in L^\infty$ converges in the norm of L^∞, then (exercise) the limit agrees with a continuous function a.e. (and the convergence of the Fourier series to that continuous function is uniform on the unit circle (or, equivalently, on the interval $[0, 2\pi]$). Thus the Fourier series of any "essentially discontinuous" function $f \in L^\infty$ cannot converge to f in the metric of L^∞. [12]

[12] *Exercise*. It can't converge (to anything) in the metric of L^∞.

LEFT OPEN, however, is the following question:

> *Suppose* $1 < p < \infty$ *and* $p \neq 2$*. Does the Fourier series of each* $f \in L^p$ *converge to* f*?*

We'll see in Section 7 that the Riesz transform provides the answer. However, right now the task is to use Fourier series to identify the action of the unique continuous extension of the Riesz transform from the space of trigonometric polynomials to L^p for $1 < p < \infty$. The key to this lies in the following

Proposition 5.1. *For each* $p \in [1, \infty)$ *and* $n \in \mathbb{Z}$*, the "n-th Fourier coefficient functional"*

$$f \to \widehat{f}(n) \qquad (f \in L^p)$$

is continuous on L^p*.*

Proof. The functional is clearly linear, so we need only prove its continuity at the origin. This comes from Hölder's inequality:

$$|\widehat{f}(n)| = \left| \int f \, e_{-n} \right| \leq \|f\|_p \|e_{-n}\|_{p'} \leq \|f\|_p \,,$$

where p' is the Hölder conjugate index of p,[13] and $\|e_n\|_{p'} = 1$ because $|e_n(e^{i\theta})| = |e^{in\theta}| = 1$ for every θ. This establishes the desired continuity. □

[13] i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

Now we can identify the action of the Riesz transform R on L^p for $2 \leq p < \infty$. Fix $f \in L^p$ and choose a sequence (f_k) of trigonometric polynomials that converges to f in L^p-norm. Then for each $n \in \mathbb{Z}$, we know from Proposition 5.1 that for each $n \in \mathbb{Z}$:

$$\widehat{Rf}(n) = \lim_k \widehat{Rf_k}(n) = \lim_k \omega_n \widehat{f_k}(n) = \omega_n \widehat{f}(n) \,,$$

where, as usual, the "multiplier" ω_n is defined by (10). Thus we've proved

Theorem 5.2. *Suppose* $2 \leq p < \infty$*. If* $f \in L^p$ *then* Rf *is the function in* L^p *with Fourier series*

$$\sum_{n \in \mathbb{Z}} \omega_n \widehat{f}(n) e^{in\theta} \,. \tag{14}$$

6 *The Riesz transform on L^p for $1 < p \leq 2$.*

OUR EXTENSION of the inequalities (R_{2n}) to (R_p) for all $p \in (2, \infty)$ depended on the Riesz-Thorin Theorem, the proof of which Littlewood called "the most impudent method in mathematics." In this section we'll extend the result to $1 < p < 2$, by what must therefore the the "second-most impudent method in mathematics," i.e., the *Method of Duality*.

Definition 6.1. If X and Y are Banach spaces, with dual spaces X^* and Y^* respectively, then for each a continuous linear transformation $T: X \to Y$ the formula

$$(T^* y^*)(x) = y^* T(x) \qquad (y^* \in Y^*,\ x \in X)$$

defines a continuous linear transformation $Y^* \to X^*$; it is called the *adjoint* of T.

In our case, $X = Y = L^p$ for some value of p strictly between 1 and 2, and—thanks to the Riesz Representation Theorem—we can represent X^* and Y^* by $L^{p'}$, with $p' \in (2, \infty)$ the Hölder conjugate index, and each vector $g \in L^{p'}$ inducing continuous linear functional Φ_g on L^p via

$$\Phi_g(f) = \int f\, g \qquad (f \in L^p)\,. \tag{15}$$

With this identification of the dual space, the Riesz transform, which we know is continuous on $L^{p'}$ has its adjoint R^* continuous on L^p.

WHAT IS $R^*: L^p \to L^p$ for $1 < p < 2$?

To find out, fix trigonometric polynomials $f = \sum \widehat{f}(n) e^{in\theta}$ and $g = \sum \widehat{g}(n) e^{in\theta}$. We know from Definition 6.1, the duality pairing (15), and the orthogonality relations (3) that

$$\Phi_g(f) = \sum_n \widehat{f}(n)\, \widehat{g}(-n)\,. \tag{16}$$

hence

$$\Phi_{R^* f}(g) = \sum_n \omega_{-n} \widehat{f}(n) \widehat{g}(-n) = \Phi_{-Rf}(g)\,.$$

Thus $R^* = -R$ on the vector space $\mathscr{T}$ of trigonometric polynomials, so by the continuity of both R and R^* and the density of $\mathscr{T}$ in the spaces L^p spaces under consideration here, we have $R^* = -R$ on all of L^p.

CONCLUSION: For $1 < p < 2$ the Riesz transform R extends continuously from the space of trigonometric polynomials to all of L^p, and this extension is given, just as in the case $2 \leq p < \infty$, by the "Fourier multiplier equation" (14).

7 Mean convergence of Fourier Series

SO FAR we have shown that if $1 < p < \infty$ and ω_n as given by (10) above, then:

- *For each $f \in L^p$, with Fourier series $\sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{in\theta}$, the "conjugate series"*
$$\sum_{n\in\mathbb{Z}} \omega_n \widehat{f}(n)e^{in\theta}$$
is the Fourier series of an L^p function Rf. Moreover,
- *The "Riesz transform" $f \to Rf$ is a continuous linear mapping on L^p.*

In this section we'll use the Riesz transform to study the question of whether or not the Fourier series of an L^p function converges, in L^p, to that function. More precisely, for $f \in L^1$, define

$$S_N f := \sum_{|n|\leq N} \widehat{f}(n)e^{in\theta},$$

the "N-th symmetric partial sum of the Fourier series of f." Convergence of this Fourier series to f, in whatever mode is being discussed at the time, always refers to convergence of $S_N f$ to f as $N \to \infty$.

As mentioned earlier: for $f \in L^2$, Hilbert-space theory and density of the trigonometric polynomials in L^2 imply that the Fourier series of f converges to f in L^2. For other values of $p \in (1,\infty)$ we still have the L^p-density of trigonometric polynomials, but we do not have the notion of orthogonality that underlies the Hilbert-space argument. In its place we have the continuity of the Riesz transform, and this turns out to be all that's needed.

For $1 < p < \infty$, the L^p-continuity of the Riesz transform guarantees that the linear mapping P defined by

$$Pf := \frac{1}{2}(f + Rf) + \frac{1}{2}\widehat{f}(0) \qquad (f \in L^p).$$

takes L^p continuously into itself; by Theorem 5.2 its image is precisely H^p, the subspace of L^p consisting of functions f with $\widehat{f}(n) = 0$ for each $n < 0$. [14] Moreover, P is the identity map on H^p, i.e., it's a *projection* of L^p onto H^p. In honor of its connection with the Riesz transform, P is called the *Riesz Projection* of L^p onto H^p.

[14] Equivalently, H^p is the space of L^p functions whose Fourier series are "of power-series type", i.e. have the form $\sum_{n\geq 0} \widehat{f}(n)e^{in\theta}$.

Theorem 7.1. *If $1 < p < \infty$ and $f \in L^p$, then $S_N f \to f$ in L^p, i.e., the Fourier series of f converges to f in the metric of L^p.*

Proof. For $n \in \mathbb{Z}$ let M_n denote the operator of "multiplication by $e^{in\theta}$:

$$M_n f(e^{i\theta}) = e^{in\theta} f(e^{i\theta}) \qquad \theta \in \mathbb{R}.$$

Note that M_n is a linear isometry on L^p, and that the Fourier series of $M_n f$ is just

$$\sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{i(k+n)\theta} = \sum_{k\in\mathbb{Z}} \widehat{f}(k-n)e^{ik\theta},$$

i.e., the graph of the Fourier transform $\widehat{M_n f}$ is just the graph of $\widehat{f}$ translated n units to the right.

The "complementary projection" to $I - P$ projects L^p continuously onto the subspace of L^p whose positively indexed Fourier coefficients are all zero, so the mapping $Q : L^p \to L^p$ defined by

$$Qf = (I-P)f + \widehat{f}(0) \qquad (f \in L^p)$$

projects L^p continously onto the space consisting of functions whose Fourier non-negatively indexed Fourier coefficients are all zero.

I leave it to you to check (best done by drawing pictures of Fourier-transform graphs) that for each $f \in L^p$:

$$S_n f = M_n\, Q\, M_{2n}\, P M_n\, f,$$

hence

$$\|S_n f\|_p \le \|Q\|\|P\|\|f\|_p = C\|f\|_p \tag{17}$$

where $C = \|Q\|\|P\| = \|P\|\|I-P\|$ is finite, and independent of f.

Now we're in a standard end-game position. We know that $S_n f \to f$ in L^p on a dense subset (the trigonometric polynomials), and we know that the sequence of operator-norms $\|S_n\|$ is bounded (by C). Thus, given $f \in L^p$ and $\varepsilon > 0$, choose a trigonometric polynomial g for which $\|f-g\|_p < \varepsilon/(C+1)$, and choose $N > 0$ such that $n \ge N$ implies that $S_N g = g$ (i.e., choose $N >$ to be the "order" of g. Thus N depends only on ε). Then for $n > N$

$$S_n f - f = (S_n f - S_n g) + \underbrace{(S_n g - g)}_{=0} + (g - f)$$

so

$$\begin{aligned} \|Snf - f\|_p &\le \|S_n f - S_n g\|_p + \|g - f\|_p \\ &= \|S_n(f-g)\|_p + \|f-g\|_p \\ &\le C\|f-g\|_p + \|f-g\|_p \\ &< (C+1)\varepsilon/(C+1) = \varepsilon \end{aligned}$$

which completes the proof. □

8 What about $p = \infty$ and $p = 1$?

So far we've seen that for $1 < p < \infty$:

- *The* Riesz transform *maps L^p continuously into itself,*
- *Hence so does the* Riesz Projection*, and consequently*:
- *The* Fourier series *of each $f \in L^p$ converges to f in the L^p norm.*

Theorem 8.1. *The Riesz transform does not map L^∞ into itself.*

Proof. Consider the geometric series :

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1)$$

with the series on the right converging uniformly on compact subsets of the open unit disc. Upon integrating both sides of this equation on the line segment from 0 to z, and using uniform convergence to interchange the integral and sum on the right, we obtain

$$\log\left(\frac{1}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1}{n} z^n \qquad (|z| < 1), \tag{18}$$

again with the series on the right converging uniformly on compact subsets of $\{|z| < 1\}$. Here the logarithm on the left is the *principal branch* of the complex logarithm.[15]

It's an easy exercise to check that $1/(1-z)$ has real part $> 1/2$ whenever $|z| < 1$, so the principal value of its argument lies in the interval $(-\pi/2, \pi/2)$. Hence

$$F(z) := -i \log\left(\frac{1}{1-z}\right) = i \underbrace{\ln|1-z|}_{:=v(z)} + \underbrace{\arg\left(\frac{1}{1-z}\right)}_{:=u(z)}. \tag{19}$$

is analytic in the open unit disc $\mathbb{U} := \{|z| < 1\}$, so its real part u and its imaginary part v are harmonic there, with v the harmonic conjugate of u (for which $v(0) = 0$).

The crucial points here are that on the unit circle:

- *$F \in L^2$ (exercise), hence the same is true of u and v.*
- *u is* bounded*, while v, is* not*!*
- *$v = Ru$.*

Conclusion: $R(L^\infty) \not\subset L^\infty$. □

[15] Defined for $z = re^{i\theta} \in \mathbb{C}\setminus(-\infty, 0)$ by

$$\log(z) = \ln r + i\theta$$

where θ is the *principal value* of $\arg z$ (the one for which $|\theta| < \pi$).

Corollary 8.2. *The Riesz transform does not map L^1 into itself.*

Proof. Suppose R did map L^1 into itself. Then it would have to do so continuously[16], so the adjoint map R^* would map L^∞ continuously into itself. But just as in §6, we'd then have $R^* = -R$, hence the Riesz transform would take L^∞ into itself—contradicting the conclusion of Theorem 8.1 above. □

[16] By the Closed-Graph Theorem; see, e.g., [7], Ch. 5, Exercise 16, page 114.

Corollary 8.3. *There exists $f \in L^1$ whose Fourier series does not converge in the norm of L^1.*

Proof. Suppose, for the sake of contradiction, that the Fourier series of each $f \in L^1$ were to converge in L^1. Then for each $f \in L^1$, the symmetric-partial-sum sequence $(s_n f,\, n \in \mathbb{N})$ would be bounded in L^1-norm, so by the Uniform Boundedness Principle[17] there would exist a positive constant A such that

[17] See, e.g., [7], Chapter 5, Theorem 5.8, pp. 98-99 (where it's called the "Banach-Steinhaus Theorem").

$$\|s_n f\|_1 \le A\|f\|_1$$

for every $f \in L^1$ and every $n \in \mathbb{N}$.

For $N \in \mathbb{N}$ let $\mathscr{T}_{2N}$ denote the family of "trigonometric polynomials of order $\le 2N$," i.e., functions of the form

$$f(e^{i\theta}) = \sum_{-2N}^{2N} \widehat{f}(n)e^{in\theta}\,.$$

As in the proof of Theorem 7.1, let M_N denote "multiplication by the function $e^{iN\theta}$. I leave it to you to check that on $\mathscr{T}_{2N}$ the Riesz projection P is represented in terms of S_N by:

$$P|_{\mathscr{T}_{2N}} \;=\; M_N\, s_N\, M_{-N}$$

Thus for each $N \in \mathbb{N}$ and $f \in \mathscr{T}_{2N}$ we have (since $M_{\pm N}$ is an L^1-isometry)

$$\|Pf\|_1 \le A\|f\|$$

so this inequality is true for every trigonometric polynomial, and so—by the density of trigonometric polynomials in L^1—is true for each $f \in L^1$.

Conclusion: If the Fourier series of each function in L^1 were to converge in the L^1-metric, then the Riesz projection would map L^1 continuously into itself, contradicting Corollary 8.2. □

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