The Radon-Nikodym Theorem "Made Easy"

Joel H. Shapiro

May 6, 2018

Notes for a talk in the Analysis Seminar at Portland State University on von Neumann's derivation of the Radon-Nikodym Theorem from the Riesz Representation Theorem in Hilbert Space.

1 *Introduction*

The Radon-Nikodym Theorem concerns two measures μ and ν defined on a sigma-algebra $\mathscr{F}$ of subsets of a set Ω. For simplicity we'll assume both measures are *positive* and *finite*; more general situations follow directly from this special case[1].

[1] More details in §4 below.

Definition 1. *To say that "ν is* absolutely continuous *with respect to μ" (notation: $\nu \ll \mu$) means that whenever $E \in \mathscr{F}$ has μ-measure zero, it also has ν-measure zero.*[2]

[2] For finite, positive measures there is an equivalent "$\delta - \varepsilon$" definition; see [4, Th. 6.11, p. 124], for example.

For example: suppose $h \in L^1(\mu)$ is takes non-negative values a.e. $[\mu]$, and that ν is defined by:

$$\nu(E) = \int_E h\, d\mu \qquad (E \in \mathscr{F}). \tag{1}$$

Then ν is a finite positive measure on $\mathscr{F}$, and clearly $\nu \ll \mu$. The Radon-Nikodym Theorem asserts that the converse is true:

The Radon-Nikodym Theorem. *If $\nu \ll \mu$ then* (1) *holds for some μ-integrable function h on Ω that takes non-negative values a.e.$[\mu]$.*

Radon [3 (1913)] proved the theorem for μ = Lebesgue measure on $\mathbb{R}^n$, and Nikodym [2 (1930)] extended it to the general case.

A little exercise in measure theory shows that the non-negativity of h follows from that of ν (hence the finiteness of ν guarantees the integrability of h), and that h is unique a.e. $[\mu]$.[3] The function h promised by the theorem is called the *Radon-Nikodym derivative of ν with respect to μ*, often written $h = \frac{d\nu}{d\mu}$. This allows (1) to be written in the satisfying, but mystifying, form:

$$\nu(E) = \int_E \frac{d\nu}{d\mu}\, d\mu \qquad (E \in \mathscr{F}).$$

[3] "a.e. $[\mu]$" means "except on a set of μ-measure zero. So this last phrase means that if both h_1 and h_2 satisfy the conclusion of the Radon-Nikodym Theorem, then $h_1 = h_2$, except possibly on a μ-null set.

2 von Neumann's proof of the Radon-Nikodym Theorem

See [1, 1940], pp. 124–130.

To produce the desired Radon-Nikodym derivative, von Neumann called upon the

The Riesz Representation Theorem. *If Λ is a bounded linear functional on the Hilbert space H, then there exists a unique vector $v \in H$ such that*

$$\Lambda(x) = \langle x, v \rangle \qquad (x \in H). \tag{2}$$

In our case the Hilbert space H will be $L^2(\mu + \nu)$, on which we'll define Λ by:

$$\Lambda(f) = \int f\, d\nu \qquad (f \in L^2(\mu + \nu)). \tag{3}$$

To see that Λ is defined on $L^2(\mu)$, and is a bounded linear functional thereon, fix $f \in L^2(\mu + \nu)$, and use the Cauchy-Schwarz inequality on $|f|$:

$$\begin{aligned} \int |f|\, d\nu = \int |f| \cdot 1\, d\nu &\leq \left(\int |f|^2\, d\nu \right)^{\frac{1}{2}} \left(\int 1^2\, d\nu \right)^{\frac{1}{2}} \\ &= \nu(\Omega)^{\frac{1}{2}} \left(\int |f|^2\, d\nu \right)^{\frac{1}{2}} \\ &\leq \nu(\Omega)^{\frac{1}{2}} \left(\int |f|^2\, d(\mu + \nu) \right)^{\frac{1}{2}} \end{aligned}$$

Thus f is integrable with respect to ν, so (3) defines a linear functional Λ on $L^2(\mu + \nu)$, and

$$|\Lambda(f)| \leq \nu(\Omega)^{\frac{1}{2}} \, \|f\|_{L^2(\mu+\nu)},$$

i.e., Λ is a bounded linear functional on $L^2(\mu + \nu)$ (with $\|\Lambda\| \leq \nu(\Omega)^{\frac{1}{2}}$).

The Riesz Representation Theorem now supplies $g \in L^2(\mu + \nu)$ such that for each $f \in L^2(\mu + \nu)$ we have

$$\Lambda(f) = \int f\, g\, d(\mu + \nu)$$

an equation we'll employ in two equivalent forms (still valid for each $f \in L^2(\mu + \nu)$):

$$\int f\, d\nu = \int f\, g\, d(\mu + \nu), \tag{4a}$$

$$\int f\,(1 - g)\, d\nu = \int f\, g\, d\mu. \tag{4b}$$

Here's a quick "proof" of the Radon-Nikodym Theorem. For $E \in \mathscr{F}$ use Eqn. (4b) with

$$f = \chi_E \cdot \frac{g}{1-g}\,. \tag{5}$$

The result is:

$$\nu(E) = \int_E \frac{g}{1-g}\,d\mu\,, \tag{6}$$

which "proves" the theorem with

$$h = \frac{g}{1-g}\,. \tag{7}$$

Why all the quotation marks? *Answer*: Something is missing from this argument. The problem is that in order to apply Eqn. (4b) we must know that the function defined by (5) lies in $L^2(\mu+\nu)$; something upon which the Riesz Representation Theorem maintains a stony silence.

Thus we need to better understand the "Riesz function" g. Let's start by studying

- its "negative set" $N := \{g < 0\}$, and
- its "bad set" $B := \{g > 1\}$.

These two sets are disjoint, and thanks to the measurability of g, both belong to $\mathscr{F}$.

Apply (4a) with $f = \chi_N$ (clearly in $L^2(\mu+\nu)$ by the finiteness of both measures); the result is:

$$\nu(N) = \int_N g\,d(\mu+\nu) = \int_N g\,d\mu + \int_N g\,d\nu$$

The left-hand side of this equation is ≥ 0, but—since $g \leq 0$ on N (and the measures μ and ν are positive), the right-hand side is ≤ 0. Using this and (once more) the non-positivity of g on N, we conclude that $(\mu+\nu)(N) = 0$, i.e., that $\mu(N) = \nu(N) = 0$.

What about the "bad" set B? Upon setting $f = \chi_B$ in (4b), we see that

$$\int_B (1-g)\,d\nu = \int_B g\,d\mu\,.$$

Since $g > 1$ on B, the left-hand side of this equation is ≤ 0, while the right-hand side is ≥ 0. Thus both sides $= 0$; in particular:

$$0 \leq \mu(B) \leq \int_B g\,d\mu = 0,$$

so $\nu(B) = 0$, and since $1-g$ is < 0 on B we see from $\int_B (1-g)\,d\nu = 0$ that also $\nu(B) = 0$.

SO FAR: Both of the sets N and B are negligible with respect to the measure $\mu + \nu$, so we're free to re-define g as we wish on them. For definiteness, let's make $g \equiv 0$ on N and $\equiv 1$ on B. Then for our re-defined function g:

> $0 \le g(x) \le 1$ *at every point* $x \in \Omega$, *and equations* (4a) *and* (4b) *above continue to hold for each* $f \in L^2(\mu + \nu)$.

Here's another pair of important sets associated with g.

- The "good set:" $G := \{0 \le g < 1\}$,
- and "singular set:" $S := \{g = 1\}$.[4]

[4] Terminology to be explained shortly.

The sets S and G are disjoint, and since $0 \le g \le 1$ at each point of Ω, their union is all of Ω. In other words:

> *The sets G and S form a* partition *of* Ω.

The "good set" is an increasing union of even better sets:

$$G = \bigcup_{n=1}^{\infty} G_n \quad \text{where} \quad G_n := \left\{0 \le g < 1 - \frac{1}{n}\right\}.$$

Fix an index n and a set $E \in \mathscr{F}$. Let

$$f = \frac{\chi_{G_n \cap E}}{1 - g}$$

Since $1 - g$ is bounded below (by $1/n$) on G_n, the function f is non-negative and bounded *above* (by n) on $G_n \cap E$. In particular, $f \in L^2(\mu + \nu)$, so we can use it in Equation (4b). There results:

$$\nu(G_n \cap E) = \int_E \frac{\chi_{G_n} g}{1 - g}\, d\mu\,. \tag{8}$$

Let n tend to ∞ and use the monotone convergence theorem to conclude that for each $E \in \mathscr{F}$:

$$\nu(G \cap E) = \int_{G \cap E} \frac{g}{1 - g}\, d\mu. \tag{9}$$

In other words, we've proved, for the "restrictions of μ and ν to G," the Radon-Nikodym Theorem, with $h = g/(1 - g)$.

What about Ω? Recall that Ω is the disjoint union of G and S, where $S = \{g = 1\}$. Upon setting $f = \chi_S$ in (4b) we conclude that $0 = \mu(S)$. Because $\nu \ll \mu$ (*) we also have $\nu(S) = 0$, hence we can replace G by Ω on both sides of (9) to obtain the desired conclusion: *For each* $E \in \mathscr{F}$,

(*) The first time we've used this hypothesis!

$$\nu(E) = \int_E h\, d\mu \quad \text{with} \quad h := \frac{g}{1 - g}.$$

This completes the proof of the Radon-Nikodym Theorem (for finite, positive measures).

3 *The Lebesgue-Radon-Nikodym Theorem*

Suppose we have two positive, finite measures μ and ν on $\mathscr{F}$, but we do not assume that ν is absolutely continuous with respect to μ. Then the argument of the previous section shows that

$$\nu(E) = \int_E h\,d\mu + \nu(E \cap S) \qquad (E \in \mathscr{F}), \tag{10}$$

where $\mu(S) = 0$. This conclusion is often called the *Lebesgue-Radon-Nikodym Theorem*.

On the right-hand side of (10): upon writing the first summand as $\nu_a(E)$, and the second one as $\nu_s(E)$ we can rewrite that equation as

$$\mu = \nu_a + \nu_s\,, \tag{11}$$

where (with respect to μ): ν_a is the part of μ that is *absolutely continuous*, and ν_s is the part that is *singular*—meaning that it gives full measure to a set of μ-measure zero.[5]

[5] Informally: ν_s is *supported* on a set of μ-measure zero, namely the "singular set" S.

Example. Let μ be the measure that gives each Borel subset of $[0,1]$ its Lebesgue measure if the subset does not contain the origin, and its Lebesgue measure plus 1 otherwise. Then on the right-hand side of (11) we have ν_a = Lebesgue measure, and ν_s = the unit mass at the origin.

4 *Beyond finiteness*

To say a positive measure on $\mathscr{F}$ is "sigma-finite" means that the underlying set Ω can be partitioned into a countable collection of subsets of $\mathscr{F}$, each of which has finite μ-measure. For example: Lebesgue measure on $\mathbb{R}^n$ is sigma-finite. It's an easy exercise to check that the Radon-Nikodym Theorem can be extended to measures μ and ν that are merely sigma-finite—at the cost of giving up the integrability of the Radon-Nikodym derivative $f = \frac{d\nu}{d\mu}$ should $\nu(\Omega) = \infty$. Once this is done, the Hahn decomposition of a "real-valued measure" into positive and negative parts extends the theorem to all such measures, and the decomposition of a "complex" measure into real and imaginary parts takes matters even further.[6].

[6] For the details, see e.g., [4], Theorem 6.10, pp. 121–124, from which the work of §2 was adapted.

However, some extra hypothesis is needed on the measures in question. For example, suppose $\mathscr{F}$ is the collection of Lebesgue measurable subsets of $\Omega = [0,1]$, and on $\mathscr{F}$:

> ν is Lebesgue measure, while μ is *counting measure*.[7]

[7] That is, μ is the measure on $\mathscr{F}$ that assigns mass 1 to *each point* of the unit interval.

Thus ν is finite and absolutely continuous with respect to μ (since $\mu(E) = 0$ iff E is the empty set!). However there is no $h \in L^1(\mu)$ such that the Radon-Nikodym equation (1) holds. Indeed, necessary for a function h can belong to $L^1(\mu)$ is that it take non-zero values on at most a countable set, so for this situation any measure ν defined by (1) must give measure zero to all but an at most countable subset of the unit interval.

The problem is, of course, that in this example, μ is not sigma-finite! On the other hand, it *is* possible to extend the Radon-Nikodym Theorem beyond sigma-finiteness. For this see [5, §5.4]. [8]

[8] A preprint of this book is freely available at: https://people.math.ethz.ch/salamon/PREPRINTS/measure.pdf (h/t to Michael Greinecker, for his post on mathoverflow.net).

References

1. John von Neumann, *On rings of operators III*, Annals of Mathematics 41 (1940).

2. Otton Nikodym, *Sur une généralisation des intégrales de M. J. Radon*, Fundamenta Mathematicae 15 (1930) 131–179.

3. Johann Radon, *Theorie und Anwendugen der absolut additiven Mengenfunktionen*. S.-B. Akad. Wiss., Wien 122 (1913) 1295–1438.

4. Walter Rudin, *Real and Complex Analysis, 3rd ed.*, McGraw-Hill 1978.

5. Dietmar A. Salamon, Measure and Integration, EMS Textbooks in Mathematics, European Mathematical Society, 2018.