The Volterra Operator

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$$Vf(x) = \int_0^x f(t)\,dt$$

Volterra's grasp,
Integral operator flows,
Chaos tamed with grace.

Haiku composed by ChatGPT

Introduction

Setting: $L^2 =$ all (a.e. equiv. classes of) Leb. measurable functions $f\colon [0,1] \to \mathbb{C}$ such that

$$\|f\|^2 := \int_0^1 |f(x)|^2 dx < \infty$$

L^2 is a *Hilbert space* with inner product

$$\langle f,\, g\rangle := \int_0^1 f(x)\overline{g(x)}\, dx \qquad (f, g \in L^2)$$

$$\therefore \qquad \|f\|^2 = \langle f,\, f\rangle \qquad (f \in L^2)$$

The Volterra Operator

$$Vf(x) := \int_0^x f(t)\, dt \quad (f \in L^2,\ 0 \le x \le 1)$$

Vf SOLVES the initial value problem

$$y' = f(x) \text{ a.e.,}\ \ y(0) = 0.$$

Lemma. $\forall f \in L^2$ & $0 \le x_1 \le x_2 \le 1$:

$$|Vf(x_2) - Vf(x_1)| \le \sqrt{x_2 - x_1}\, \|f\|$$

Proof. $$|Vf(x_2) - Vf(x_1)| = \left|\int_{x_1}^{x_2} f(t)\, dt\right|$$

$$= |\langle \chi_{[x_1,x_2]},\, f\rangle| \ \le\ \|\chi_{[x_1,x_2]}\|\, \|f\|$$

$$= \sqrt{x_2 - x_1}\, \|f\|$$

Corollary. $V(L^2) \subsetneq C([0,1]) \subsetneq L^2$

$V(L^2)$

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The Volterra Operator

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Lemma. $\forall f \in L^2$ & $0 \le x_2 \le x_2 \le 1$:

$$|Vf(x_2) - Vf(x_1)| \le \sqrt{x_2 - x_1}\, \|f\|$$

Corollary. $V(L^2) \subsetneq C([0,1]) \subsetneq L^2$

Prop. [V a bounded operator on L^2]

$$\|Vf\| \le \tfrac{1}{\sqrt{2}}\, \|f\| \quad (f \in L^2)$$

Proof. For $f \in L^2$ and $0 \le x \le 1$:

$$|Vf(x)| \le \sqrt{x}\, \|f\|$$

$$\therefore \quad \|Vf\|^2 \le \left(\int_0^1 x\, dx\right) \|f\|^2 = \frac{1}{2}\, \|f\|^2 \qquad \square$$

Corollary. $\|V\| := \sup_{\|f\| \le 1} \|Vf\| \le \dfrac{1}{\sqrt{2}}$.

The Volterra Operator

$$Vf(x) := \int_0^x f(t)\,dt \qquad (f \in L^2)$$

So far we know:

(a) V is a bounded operator on L^2, with *operator norm* $\|V\| \leq \frac{1}{\sqrt{2}}$.

(b) $V(L^2) \subsetneq C([0,1]) \subsetneq L^2$.

Prop. *V is 1-1.*

Proof. Suppose $Vf = 0$ for some $f \in L^2$.

$$\therefore \quad 0 \underbrace{=}_{\forall x} \tfrac{d}{dx} Vf(x) \underbrace{=}_{\text{a.e.}\,x} f(x).$$

$\therefore \quad f = 0$ a.e. on $[0,1]$, i.e. $f = 0$ in L^2.

Conclude: V is 1-1. □

Theorem. *V has no eigenvalues (!)*

Proof. Suppose:

(*) $\qquad Vf = \lambda f$

for some $\lambda \in \mathbb{C}$ and $f \in L^2 \backslash \{0\}$

Then: $\lambda \neq 0$ (since V is 1-1).

Observe: By (*) (and $\lambda \neq 0$) we know $f \in C([0,1])$, so can diff both sides of (*) and use the Fund'l Thm of Integral Calculus to obtain:

$$f = \lambda f', \text{ so } f(x) = f(0)e^{x/\lambda} \text{ on } [0,1].$$

But by (*): $\quad \lambda f(0) = Vf(0) = 0 \quad$ (!!)

CONCLUDE: $f = 0$, a contradiction. □

Adjoint

$$Vf(x) := \int_0^x f(t)\,dt \qquad (f \in L^2)$$

So far, on L^2:

(a) V is a bounded operator, with

(b) $\|V\| \leq \frac{1}{\sqrt{2}}$.

(c) V is one-to-one on L^2.

(d) V has no eigenvalues .

Definition [Adjoint of V]. The linear map V^* on L^2 defined by:

(*) $\quad \langle f, V^*g\rangle := \langle Vf, g\rangle \quad (f, g \in L^2)$.

WHAT DOES THIS MEAN?

Write $\Lambda_g(f) := \langle f, g\rangle$.

Then Λ_g a bounded linear functional on L^2

... with $\|\Lambda_g\| = \|g\|$

So (*) becomes:

$$\Lambda_g(Vf) = \Lambda_{V^*g}(f)$$

That is:

$$\Lambda_g \circ V = V^*(\Lambda_g)$$

i.e., *V^* maps the bndd lin fnl Λ on L^2 to the bndd lin fnl $\Lambda \circ V$.*

Adjoint

$$Vf(x) := \int_0^x f(t)\,dt \qquad (f \in L^2)$$

So far, we've shown that on L^2:

(a) V is a bounded operator, with

(b) $\|V\| \le \frac{1}{\sqrt{2}}$

(c) V is one-to-one.

(d) V has no eigenvalues .

(e) We've defined V^* on L^2 by:

$$\langle Vf,\, g\rangle := \langle f,\, V^*g\rangle$$

Theorem. For $g \in L^2$ and $0 \le x \le 1$:

$$V^*g(x) := \int_x^1 g(t)\,dt$$

Proof. Fix $f, g \in L^2$. By definition:

$$\langle f,\, V^*g\rangle = \langle Vf,\, g\rangle = \int_0^1 (Vf)(x)\,\overline{g(x)}\,dx$$

$$= \int_{x=0}^1 \left(\int_{y=0}^x f(y)\,dy\right)\overline{g(x)}\,dx$$

$$= \int\int_{\{0\le y\le x\}} f(y)\overline{g(x)}\,dy\,dx$$

$$= \int_{y=0}^1 f(y)\overline{\left(\int_{x=y}^1 g(x)\,dx\right)}\,dy$$

$$= \langle\ f\ ,\ (\ \cdot)\ \rangle$$

$\square$

The Norm

$$Vf(x) := \int_0^x f(t)\,dt \qquad (f \in L^2),$$

So far: we have proved that on L^2:

(a) V is a compact operator.

(b) V is one-to-one (but not "onto").

(c) V has no eigenvalues.

(d) $\|V\| \le \frac{1}{\sqrt{2}}$

(e) $\langle Vf, g\rangle = \langle f, V^*g\rangle$ (defn. of V^*)

(f) $V^*g(x) = \int_x^1 g(t)\,dt \quad (0 \le x \le 1)$

Exercise. V^* is unitarily equivalent to V.

Prop. $\frac{2}{\pi} \le \|V\| \le \frac{1}{\sqrt{2}}$

Proof (of "$\le$"). For $w(x) := \cos(\frac{\pi}{2}\,x)$:

(**) $\quad Vw(x) = \frac{2}{\pi}\sin(\frac{\pi}{2}\,x)$,

$\therefore \quad \|Vw\| = \frac{2}{\pi}\|w\|$. □

Prop. The function $w(x) = \cos(\frac{\pi}{2}\,x)$ is an *eigenfunction* of V^*V:

$$(V^*V)w = \left(\tfrac{2}{\pi}\right)^2 w$$

Proof. Apply V^* to both sides of (**):

$$(V^*Vw)(x) = \left(\frac{2}{\pi}\right)^2 \underbrace{\cos\left(\frac{\pi}{2}\,x\right)}_{=w(x)}$$ □

The Norm

$$Vf(x) := \int_0^x f(t)\,dt \qquad (f \in L^2),$$

So far: we have proved that on L^2:

(a) V is a bounded operator on L^2.

(b) V is one-to-one (but not "onto").

(c) V has no eigenvalues.

(d) $\frac{2}{\pi} \le \|V\| \le \frac{1}{\sqrt{2}}$

(e) $V^*g(x) = \int_x^1 g(t)\,dt \quad (0 \le x \le 1)$

(f) $(V^*V)\cos(\frac{\pi}{2}\,\cdot) = \left(\frac{2}{\pi}\right)^2 \cos(\frac{\pi}{2}\,\cdot)$

Theorem. $\|V\| = \frac{2}{\pi}$

Proof. $w(x) := \cos(\frac{\pi}{2}\,x) \ge 0$ on $[0,1]$.

POINTWISE EST. For $f \in L^2$ & $0 \le x \le 1$:

$$|(Vf)(x)| \le \int_0^x |f(t)|\,dt = \int_0^x \frac{|f(t)|}{\sqrt{w(t)}}\sqrt{w(t)}\,dt$$

$$\le \left(\int_0^x \frac{|f(t)|^2}{w(t)}\,dt\right)^{1/2} \Bigg(\underbrace{\int_0^x w(t)\,dt}_{(Vw)(x)}\Bigg)^{1/2}$$

$$\therefore \quad \|Vf\|^2 = \int_0^1 |Vf(x)|^2\,dx$$

$$\le \int_{x=0}^1 \left(\int_{t=0}^x \frac{|f(t)|^2}{w(t)}\,dt\right)(Vw)(x)\,dx$$

$$= \int_{t=0}^1 \Bigg(\underbrace{\int_{x=t}^1 (Vw)(x)\,dx}_{(V^*Vw)(t) = (\frac{2}{\pi})^2 w(t)}\Bigg)\frac{|f(t)|^2}{w(t)}\,dt$$

$$= \left(\frac{2}{\pi}\right)^2 \|f\|^2. \qquad \square$$

Summing up

So far: for $f \in L^2$ and $0 \le x \le 1$:

$$(Vf)(x) := \int_0^x f \quad \& \quad (V^*f)(x) := \int_x^1 f$$

WE HAVE PROVED that on L^2:

(*) V and V^* are a bounded operators

(*) V & V^* are one-to-one (not "onto").

(*) V and V^* have no eigenvalues.

(*) $\left(\frac{2}{\pi}\right)^2$ an eigenvalue of V^*V,

with eigenvector $\cos\left(\frac{\pi}{2}x\right)$

(*) $\|V\| = \frac{2}{\pi}$.

We have learned: In ∞-dim'l Hilbert space, bounded operators:

(*) May be one-to-one, but not onto

(*) Need not have eigenvalues

(*) Operator norms may not be obvious; their computation may not be easy.

Next Time we'll find the SVD of V by:

(*) Proving V is a *compact* operator on L^2, so V^*V is a *positive* compact operator.

(*) Finding the Spectral repn. of V^*V, hence that of $|V| := \sqrt{V^*V}$

IN ADDITION, we'll find a "closed-form" repn. of $|V|$ as an integral operator.