SVD For the Volterra Operator

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$$(Vf)(x) = \int_0^x f(t)\,dt$$

Introduction

Last time: for $f \in L^2$ and $0 \le x \le 1$ we:

- *Defined*: $(Vf)(x) := \int_0^x f(t)\,dt$

  V^* on L^2 by: $\langle f, V^*g\rangle := \langle Vf, g\rangle$
- *Proved*: $(V^*f)(x) := \int_x^1 f(t)\,dt$
- *Showed*: V and V^* are

  bounded operators,

  one-to-one (not "onto"),

  have no eigenvalues.
- *Observed*: $\left(\frac{2}{\pi}\right)^2$ an eigenvalue of V^*V,

  with eigenvector $\cos\left(\frac{\pi}{2}x\right)$
- *Proved*: $\|V\| := \sup_{\|f\|\le 1} \|Vf\| = \frac{2}{\pi}$.

Today: we'll derive:

The Volterra SVD. *There exist:*

(a) *o.n. bases* $(e_n)_1^\infty$ *and* $(h_n)_1^\infty$ *for* L^2, *&*

(b) *a positive sequence* $s_n \searrow 0$, *such that*

$$Vf = \sum_{n=1}^{\infty} s_n \langle f, e_n\rangle h_n \quad (f \in L^2),$$

the series converging in the norm of L^2.

In talks on SVD by Jim [4] and Sheldon [2], we saw that: $s_1 = \|V\|$, and more generally, s_n is the operator-norm distance from V to the set of bounded operators on L^2 having rank $< n$.

Compactness

SUPPOSE T is a bounded operator on a separable Hilbert space H.

Defn. To say "T is compact" means:

T takes each bounded subset of H into a relatively compact set.

i.e., if (f_n) is a bounded sequence in H then (Tf_n) has a (norm-) convergent subsequence.

Prop. (From last time).

For $f \in L^2$ and $0 \le x_1 \le x_2 \le 1$:

$$|Vf(x_2) - Vf(x_1)| \le \sqrt{x_2 - x_1}\,\|f\|$$

Proof. $|Vf(x_2) - Vf(x_1)| = \left|\int_{x_1}^{x_2} f(t)\,dt\right|$

$$= |\langle \chi_{[x_1,x_2]}, f\rangle| \le \|\chi_{[x_1,x_2]}\|\,\|f\|$$

$$= \sqrt{x_2 - x_1}\,\|f\| \qquad \square$$

Corollary. *V is a compact operator on L^2.*

Proof. By above Prop. and Arzela-Ascoli,

$V(B)$ is relatively compact in $C([0,1])$ for each bounded $B \subset H$.

Now $C([0,1]) \subset L^2$ and $\|\cdot\|_\infty \ge \|\cdot\|_2$

Conclude. $B \subset L^2$ bounded $\implies$ $V(B)$ relatively compact in L^2. $\square$

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For $f \in L^2$ and $0 \le x_1 \le x_2 \le 1$:

$$|Vf(x_2) - Vf(x_1)| \le \sqrt{x_2 - x_1}\, \|f\|$$

Proof. $|Vf(x_2) - Vf(x_1)| = \left| \int_{x_1}^{x_2} f(t)\, dt \right|$

$$= |\langle \chi_{[x_1, x_2]}, f \rangle| \le \|\chi_{[x_1,x_2]}\| \, \|f\|$$

$$= \sqrt{x_2 - x_1}\, \|f\| \qquad \square$$

Corollary. *V is a compact operator on L^2.*

Prop. *If T is compact and S bounded then ST and TS are compact.*

Defn. To say T is *positive*, means that

$$\langle Tf, f \rangle \ge 0 \quad \forall\, f \in H.$$

NOW RECALL THE

Defn of Adjoint. $\langle Tf, g \rangle = \langle f, T^* g \rangle$

Prop. *T^*T is positive for any op. T on H.*

Proof. $0 \le \|Tf\|^2 = \langle Tf, Tf \rangle = \langle T^*Tf, f \rangle$ $\square$

Corollary. *V^*V is positive and compact.*

Spectral Decomposition

A Spectral Theorem. Suppose the operator T is positive and compact. Then $\exists$:

(a) An orthonormal basis $(e_n)_1^\infty$ for H, and

(b) A sequence $\lambda_n \searrow 0$, such that

(*) $$Tf = \sum_{n=0}^{\infty} \lambda_n \langle f, e_n \rangle e_n \qquad (f \in H),$$

the series converging in the norm of H.

Remark. The λ_n's are the eigenvalues of T; the e_n's are the corresponding eigenvectors.

Prop. *Every positive compact operator has a (positive) square root.*

Proof. In the spectral rep'n (*) of T, replace λ_n by $\sqrt{\lambda_n}$. Resulting series converges to a bounded operator S on H with $S^2 = T$.

Theorem. (Spectral decomp. of V^*V.)

$$V^*Vf = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle e_n,$$

where $\lambda_n = ((2n-1)\frac{\pi}{2})^{-2}$,

and $e_n(x) = \sqrt{2}\cos\left((2n-1)\frac{\pi}{2}x\right)$.

Proof. Easy to check that $V^*Ve_n = \lambda_n e_n$.

Conversely, suppose $\lambda \in \mathbb{C}$ and $f \in L^2$ with

(**) $$V^*Vf = \lambda f$$

Then $\lambda \neq 0$ (V & V^* are 1-1, hence also V^*V).

Recall: For $f \in L^2$ and $0 \leq x \leq 1$:

$$Vf(x) = \int_0^x f \text{ and } V^*f = \int_x^1 f.$$

Exercise: $\lambda \neq 0$ & (**) $\implies f \in C^\infty([0,1])$

Spectral Decomposition of V^*V

Theorem. (Spectral decomp. of V^*V.)

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Recall: For $f \in L^2$ and $0 \leq x \leq 1$:

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Exercise: $\lambda \neq 0$ & (**) $\implies f \in C^\infty([0,1])$

$\therefore$ can diff. both sides of (**) twice to get

$$-f = \lambda f'' \text{ , i.e., } f'' + \tfrac{1}{\lambda} f = 0,$$

which has general solution

$$f(x) = a\cos(x/\sqrt{\lambda}) + b\sin(x/\sqrt{\lambda})$$

subject to boundary conditions

$$\lambda f(1) = V^*Vf(1) = 0, \text{ i.e., } f(1) = 0$$

and

$$\lambda f'(0) = -Vf(0) = 0, \text{ i.e., } f'(0) = 0,$$

which imply

$$f(x) = \text{const. } \cos(x/\sqrt{\lambda})$$

with

$$\lambda = \left((2n-1)\tfrac{\pi}{2}\right)^{-2} \qquad \square$$

NEXT: The SVD of the Volterra Operator

The SVD of V

Theorem. (Spectral decomp. of V^*V.)

$$V^*Vf = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle e_n,$$

where $\lambda_n = \left((2n-1)\frac{\pi}{2}\right)^{-2}$,

and $e_n(x) = \sqrt{2}\cos\left((2n-1)\frac{\pi}{2}\,x\right)$.

Defn. ("Abs. value" of V). This is the "square root" of V^*V, defined on L^2 by:

$$|V|f := \sum_{n=1}^{\infty} \sqrt{\lambda_n} \langle f, e_n \rangle e_n$$

Theorem (The SVD of V). $\forall\, f \in L^2$:

$$Vf = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \langle f, e_n \rangle h_n$$

where

$$h_n(x) = \sqrt{2}\sin\left((2n-1)\frac{\pi}{2}\,x\right)$$

Proof. SVD in infinite dimensional Hilbert space works for compact operators "just like" it does for linear operators in finite dimensions. In other words:

$$Vf = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, \langle f, e_n \rangle h_n$$

with (λ_n) and (e_n) as in Thm. at left, and

$$h_n(x) = Ve_n(x)/\sqrt{\lambda_n}$$

$$= \sqrt{2}\,\sin\left((2n-1)\frac{\pi}{2}\,x\right)$$

Theorem (Closed-form for $|V|$) For $f \in L^2$:

$$|V|f(x) = \int_{t=0}^{1} K(x,t)\, f(t)\, dt \quad (0 \le x \le 1)$$

where

$$K(x,t) = \frac{1}{\pi} \log \left| \frac{\cos\frac{\pi t}{2} + \cos\frac{\pi x}{2}}{\cos\frac{\pi t}{2} - \cos\frac{\pi x}{2}} \right| .$$

Proof of "Closed Form for $|V|$"

$$|V|f(x) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\langle f,\, e_n\rangle\, e_n(x)$$

$$= \sum_{n=1}^{\infty} \sqrt{\lambda_k}\left(\int_0^1 f(t)\, e_n(t)\, dt\right) e_n(x)$$

$$= \int_0^1 \Bigg(\underbrace{\sum_{n=1}^{\infty} \sqrt{\lambda_k}\, e_n(t)\, e_n(x)}_{K(x,t)}\Bigg) f(t)\, dt$$

Now write $e_n(t) = \sqrt{2}\cos\theta_n(t)$, where

$$\theta_n(t) = (2n-1)\tfrac{\pi}{2}\, t\,,$$

so

$$e_n(t)\, e_n(x) = 2\cos\theta_n(t)\cos\theta_n(x)$$

$$= \cos\theta_n(t+x) + \cos\theta_n(t-x)\,.$$

whereupon

$$K(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos\theta_n(t+x) + \cos\theta_n(t-x)}{2n-1}$$

After some manipulation with complex-exponential geometric series:

$$(*) \qquad \sum_{n=1}^{\infty} \frac{\cos(2n-1)\theta}{2n-1} = \frac{1}{2}\log\left|\cot\frac{\theta}{2}\right|$$

from which we obtain

$$\pi K(x,t) = \log\left|\cot\frac{\pi(t+x)}{4}\right| + \log\left|\cot\frac{\pi(t-x)}{4}\right|$$

$$= \log\left|\cot\frac{\pi(t+x)}{4}\,\cot\frac{\pi(t-x)}{4}\right|$$

$$= \log\left|\frac{\cos\frac{\pi(t+x)}{4}\,\cos\frac{\pi(t-x)}{4}}{\sin\frac{\pi(t+x)}{4}\,\sin\frac{\pi(t-x)}{4}}\right|$$

$$= \log\left|\frac{\cos\frac{\pi t}{2} + \cos\frac{\pi x}{2}}{\cos\frac{\pi t}{2} - \cos\frac{\pi x}{2}}\right|. \qquad \square$$

References

1. Sheldon Axler, *Linear Algebra Done Right, 4th ed.* SVD in Chapter 7.
2. ____________, *The Singular Value Decomposition.* Slides for two talks in the 2022-23 Portland State Univ. Analysis Seminar.
3. William Ross, *Mini-course on Operators on Function Spaces*. In: the Fields Institute's online: *Focus Program on Analytic Function Spaces and their Applications*, July–December, 2021.
4. James Rulla, *The Singular Value Decomposition for Analysts*, Slides for two talks in the 2022-23 Portland State Univ. Analysis Seminar.
5. Robert Schatten, *Norm Ideals of Completely Continuous Operators*, Springer 1960.
6. Joel H. Shapiro, *The Volterra Operator.* Slides for a talk at the 2022-23 Portland State Univ. Analysis Seminar.
7. ____________ *Volterra Variations*. Rough draft of some notes on the Volterra operator.