Recurrence in Dynamical Systems and Combinatorics

Peter Oberly

4/19/2019

Abstract

In this talk, we will illustrate a technique which connects results in additive combinatorics with statements on recurrence in dynamical systems. In particular, we will show how van der Waerden's theorem on colorings of the integers is equivalent to a multiple recurrence theorem in topological dynamics, and how Szemerédi's theorem follows from a deep generalization of the Poincaré recurrence theorem.

The relationship between dynamical systems and number theory has a long history, stretching back at least to Gauss's dynamical study of continued fractions. In these notes, we explore how theorems on arithmetic progressions are equivalent to recurrence results in both topological dynamics and ergodic theory. We begin with topological dynamics.

Definition (Topological Dynamical System)**.** Let X be a compact metric space and let $T : X \to X$ be a homeomorphism. Then the pair (X, T) is called a *topological dynamical system.*

Definition (Recurrence)**.** A point x in a TDS (X, T) is said to be a *recurrent* if for each neighborhood V of x, there is an $n \in \mathbb{N}$ so that $T^n x \in V$. Equivalently, x is recurrent if there is a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers so that $T^{n_k} x \to x$ as $n_k \to \infty$.

Example. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane. Let $\alpha \in S^1$. Then the rotation $T : S^1 \to S^1$ defined by $T(z) = \alpha z$ is a homeomorphism of S^1. So (S^1, T) is a TDS. Every point of S^1 is recurrent.

The following example of a topological system will allow us to connect arithmetic progressions to recurrence.

Example (Symbolic System)**.** Let F be a finite set. Let $F^{\mathbb{Z}}$ be the sequence space $F^{\mathbb{Z}} = \{\{x_n\}_n : x_n \in F$ for all $n \in \mathbb{Z}\}$. Define a metric d on $F^{\mathbb{Z}}$ as follows. Given $x, y \in F^{\mathbb{Z}}$, set

$$d(x, y) = \inf\{\frac{1}{k+1} : x_j = y_j, \ \ |j| < k\}$$

if $x_0 = y_0$, and instead set $d(x, y) = 1$ if $x_0 \neq y_0$. So 1 is the furthest distance apart any two elements of $F^{\mathbb{Z}}$ can be. It is worth while to examine what convergence means with respect to this metric. Suppose a sequence $\{x^{(n)}\}_{n=1}^{\infty}$ in $F^{\mathbb{Z}}$ converges to x. Then $\inf\{\frac{1}{k+1} : x_j^{(n)} = x_j \ |j| < k\} \to 0$ as $n \to \infty$; it follows that $x^{(n)}$ and x agree on greater and greater blocks centered about the 0^{th} coordinate. Put another way, $x^{(n)} \to x$ as $n \to \infty$ if and only if for each $N \in \mathbb{N}$ there is an M so that

$$x_{-N}^{(n)} = x_{-N}, ..., x_0^{(n)} = x_0, ..., x_N^{(n)} = x_N$$

for all $n \geq M$.

Now with this metric, we can show that $F^{\mathbb{Z}}$ is sequentially compact. Let $\{x^{(n)}\}_n$ be a sequence of elements in $F^{\mathbb{Z}}$. As F is a finite set, and $\{x_0^{(n)} : n \in \mathbb{N}\}$ is infinite, the pigeon hole principle implies that there are infinitely many n_i so that $x_0^{(n_i)}$ are all equal. Similarly, by the pigeon hole principle again, there are infinitely many of the $x^{(n_i)}$ which also must agree on the first and negative first coordinate. And of this subsubsequence, there are infinitely many which agree on the second and minus second coordinate. Continuing in this manner, we can create a convergent subsequence of $x^{(n)}$. So $F^{\mathbb{Z}}$ is compact.

Now there is a natural map to consider on sequence spaces: the shift map $T : F^{\mathbb{Z}} \to F^{\mathbb{Z}}$ defined by $T(..., x_{-1}, \hat{x_0}, x_1, ...) = (..., x_{-1}, x_0, \hat{x_1}, ...)$, where the ˆ denotes the zeroth coordinate. This is often abbreviated by writing $(Tx)_n = x_{n+1}$. With respect to the metric d, it is readily seen that T is a homeomorphism. For if $x^{(n)}$ and x agree on a central block of length N, then $T(x^{(n)})$ and $T(x)$ agree on a central block of length $N-1$. So $x^{(n)} \to x$ implies $T(x^{(n)}) \to T(x)$. Similarly for the inverse of T, which is a shift in the opposite direction. Hence $(F^{\mathbb{Z}}, T)$ is a topological dynamical system.

Consider what recurrence might mean in $(F^{\mathbb{Z}}, T)$. If x is recurrent, then for each open set U containing x, there is an n so that $T^n(x) \in U$. It follows that $T^{n_k}(x)$ and x agree on greater and greater central blocks; so for each N, there is an n so that

$$(T^n x)_{-N} = x_{n-N} = x_N, ..., (T^n x)_0 = x_n = x_0, ..., (T^n x)_N = x_{N+n} = x_N.$$

This form of recurrence already looks somewhat combinatorial.

Definition (Arithmetic Progression)**.** An *arithemtic progression* (AP) is a finite sequence of the form $\{a + jb\}_{j=0}^{k-1}$, where $a, b \in \mathbb{Z}$ with $b \neq 0$. The *length* of an AP is the number of terms in the sequences. The integer b is sometimes called the *gap* or *common difference*.

The following theorem, proven in 1927 by van der Waerden, is one of the first modern results on arithmetic progressions.

Theorem 0.1 (van der Waerden)**.** *Let $c : \mathbb{Z} \to F$ be a map of $\mathbb{Z}$ into a finite set F. Then for any $k \in \mathbb{N}$, there is an arithmetic progression $\{a + jb\}_{j=0}^{k-1}$ of length k so that $c(a) = c(a + b) = ... = c(a + (k-1)b)$.*

When discussing van der Waerden's theorem, the map $c : \mathbb{Z} \to F$ is referred to as a *coloring* and the fibers $c^{-1}(\{f\})$ are called *color classes.* Put another way, van der Waereden's theorem states that given any coloring of the integers, there are AP's of arbitrary length all within the same color class. Note that there are many colorings of $\mathbb{Z}$ which do not have *infinite* APs. One such is given by coloring an integer red if it has an odd number of digits, and blue if it has an even number of digits. We'll show that van der Waerden's theorem is equivalent to the following theorem on recurrence in a topological system.

Theorem 0.2 (Multiple recurrence in open covers)**.** *Let (X, T) be a topological dynamical system. Let $\{U_\alpha\}_\alpha$ be an open cover of X. Then there is a U_α so that for each $k \geq 0$, we have $U_\alpha \cap T^n U_\alpha \cap ... \cap T^{kn} U_\alpha \neq \emptyset$ for some $n > 0$.*

Theorem 0.3. *The multiple recurrence theorem is equivalent to van der Waerden's theorem.*

Proof. First assume that van der Waerden's theorem is true. Let $k \in \mathbb{N}$, and let (X, T) be a topological dynamical system. As X is compact, choose a finite subcover $\{U_1, ..., U_N\}$ of $\{U_\alpha\}_\alpha$. Choose $x \in X$. Assign a color $c(n) \in \{1, ..., N\}$ to $n \in \mathbb{Z}$ if $T^n x \in U_{c(n)}$. If n is assigned two or more colors, choose one. Then we have a map $c : \mathbb{Z} \to \{1, ..., N\}$. By van der Waerden's theorem, there is an AP $\{a, a + b, ..., a + (k - 1)b\}$ so that $c(a) = c(a + b) = ... = c(a + (k - 1)b) = r$, for some $1 \leq r \leq N$. Then $T^{a-in}x \in U_r$ for all $i = 0, ..., k$. Therefore $T^a x \in U_r \cap T^n U_r \cap ... \cap T^{kn} U_r$.

Conversely, suppose that the multiple recurrence theorem for open covers holds. The plan is to apply the multiple recurrence theorem to a subsystem of a symbolic system. Let F be a finite set of colors and let $c : \mathbb{Z} \to F$ be a coloring of the integers. Consider c as a bi-infinite sequence $c \in F^{\mathbb{Z}}$. Let X be the orbit closure of c: that is, $X = \text{cl}\ \{T^n c : n \in \mathbb{Z}\}$. Then X is compact (since $F^{\mathbb{Z}}$ is), and $T(x) \in X$ for all $x \in X$. So $(X, T|_X)$ is a TDS. For each $f \in F$, let $U_f = \{x \in X : x_0 = f\}$. Each U_f is open, and so $\{U_f\}_{f \in F}$ is an open cover of X. Therefore, by multiple recurrence in open covers, there is some color, f_0 say, so that $U_{f_0} \cap T^n U_{f_0} \cap ... \cap T^{kn} U_{f_0} \neq \emptyset$ for all $k \in \mathbb{N}$ and some n depending on k. As $U_{f_0} \cap T^n U_{f_0} \cap ... \cap T^{kn} U_{f_0}$ is open, and the iterates of c are dense in X, then there is an $m \in \mathbb{Z}$ so that $T^{-m} c \in U_{f_0} \cap T^n U_{f_0} \cap ... \cap T^{kn} U_{f_0}$. Hence $c \in T^{jn+m} U_{f_0}$ for $j = 0, ..., k$, which means that $c_{jn+m} = f_0$ for $j = 0, ..., k$. Hence the color class $c^{-1}(\{f_0\})$ has an arithmetic progression of length k for each $k \in \mathbb{N}$. □

While interesting and elegant, van der Waerden's theorem is quite qualitative in nature. It does not say which color class AP's of arbitrary length will belong to, nor how big such a subset of the integers must be. Inspired partly by van der Waerden's theorem, in 1936 Erdös and Turán conjectured that any subset $A \subset \mathbb{Z}$ which is a "positive fraction" of the integers will have AP's of any length. By positive fraction, we mean the following.

Definition (Upper Density)**.** Let $A \subset \mathbb{Z}$. The *upper density* of A is the real number

$$d^*(A) = \limsup_{N \to \infty} \frac{1}{2N + 1} |A \cap [-N, N]|.$$

If instead $A \subset \mathbb{N}$, the upper density is defined to be

$$d^*(A) = \limsup_{N \to \infty} \frac{1}{N} |A \cap [-N, N]|.$$

Remark. If we replaced the lim sup with a limit, we obtain the *natural density*, which is the asymptotic relative frequency of A in $\mathbb{Z}$. So why are we working with the lim sup? The answer is that the natural density is not always defined, and even when it is, it fails to be compatible with the Boolean structure of $\mathcal{P}(\mathbb{Z})$. For example, set $I_n = \{2^n, ..., 2^{n+1} - 1\}$. Then $A = \bigcup_n I_{2n}$ fails to have a natural density; its upper density is $2/3$ and its lower density is $\liminf \frac{1}{N} |A \cap [1, N]| = 1/3$.

So in 1936, Erdös and Turán conjectured that:

Theorem 0.4 (Szemerédi)**.** *If A is a subset of $\mathbb{Z}$ with $d^*(A) > 0$, then A has arithmetic progressions of length k for all k.*

This conjecture took nearly forty years to prove. The case for $k = 3$ was first proven by Roth in 1953. This argument did not readily generalize, and no further progress was made until Szemerédi proved the case for $k = 4$ in 1969 by different methods. The methods Szemerédi used did generalize, resulting in a proof of the full conjecture six years later (which was called a "masterpiece of combinatorial reasoning" by Erdös). Surprisingly, a proof using completely different methods was discovered by Hillel Furstenberg two years later. Furstenberg had noticed a general correspondence between theorems in combinatorics and recurrence theorems in dynamical systems (of which the above example with van der Waerden's theorem is an instance). He then used this correspondence to show that Szemerédi's theorem is equivalent to the following multiple recurrence theorem in ergodic theory:

Theorem 0.5 (Furstenberg's Multiple Recurrence). *Let $(X, \mathcal{B}, \mu)$ be a probability measure space and let $T : X \to X$ be a measure-preserving invertible transformation on $(X, \mathcal{B}, \mu)$, and let $A \in \mathcal{B}$ be a set with positive measure. Then for all $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ so that*

$$\mu\left(\bigcap_{j=0}^{k-1} T^{-jn} A\right) > 0$$

Notice that if $k = 2$, then theorem 0.5 states that $\mu(A \cap T^{-n}A) > 0$ for some n whenever $\mu(A) > 0$. This is a formulation of the Poincaré recurrence theorem. Also note that the introduction of measure preserving dynamics seems correct here; we are considering a map $d^* : \mathbb{Z} \to [0, 1]$, and would like to say that d^* is a measure on $\mathbb{Z}$ in some sense. The problem is that d^* is not a measure, and even if it were, the integers do not admit a shift invariant probability measure. So we have to be a bit more clever. We adapt the machinery from the equivalence between van der Waerden's theorem and the multiple recurrence theorem in open covers.

Let $A \subset \mathbb{Z}$, and view the characteristic function χ_A of A as an element of $2^{\mathbb{Z}} = \{0, 1\}^{\mathbb{Z}}$. Then A has an AP, say $\{a, a+b, ..., a+(k-1)b\}$, of length k if and only only $\chi_A(a) = \chi_A(a+b) = ... = \chi_A(a+(k-1)b) = 1$; the difference between this and van der Waerden's theorem is that we need to know that the coloring χ_A has an AP in a specific color class (namely, the class corresponding to 1). Now let T denote the shift map on $2^{\mathbb{Z}}$. Then $\{a + jb\}_{j=0}^{k-1} \subset A$ if and only if $(T^a\chi_A)(0) = (T^{a+b}\chi_A)(0) = ... = (T^{a+(k-1)b}\chi_A)(0) = 1$. Hence

$$\{a + jb\}_{j=0}^{k-1} \subset A \iff T^a\chi_A \in \bigcap_{j=0}^{k-1} T^{-jb}(\{x \in 2^{\mathbb{Z}} : x_0 = 1\}.$$

Now this intersection has many irrelevant points for our purposes. Like in van der Waerden's theorem, we work with a more focused subset. Let $X = \mathrm{cl}\{T^n\chi_A : n \in \mathbb{Z}\}$. As noted above, X is a compact T-invariant subset, and so $(X, T|_X)$ is a TDS. As $C_0 = \{x \in X : x_0 = 1\}$ is open in X, then if $\bigcap_{j=0}^{k-1} T^{-jb}(C_0) \neq \emptyset$, we can find an $a \in \mathbb{Z}$ so that $T^a(\chi_A) \in \bigcap_{j=0}^{k-1} T^{-jb}(C_0)$. Therefore, to show that theorem 0.5 implies Szemerédi's theorem, it suffices to show that $\bigcap_{j=0}^{k-1} T^{-jb}(C_0) \neq \emptyset$. We do so by creating a measure μ on the Borel sets $\mathcal{B}$ of X which is T-invariant and satisfies $\mu(C_0) > 0$. Then theorem 0.5 implies that $\mu(\bigcap_{j=0}^{k-1} T^{-jb}(C_0)) > 0$ and so this intersection is non-empty.

Theorem 0.6. *Theorem 0.5 implies Szemerédi's theorem.*

Proof. Assume that A has positive upper density. For convenience, set $a_n = T^n\chi_A$. For $N \geq 0$, define a sequence $\{\mu_N\}$ of probability measures on X by

$$\mu_N = \frac{1}{2N+1} \sum_{n=-N}^{n=N} \delta_{a_n}$$

where δ_{a_n} is the unit point measure concentrated at a_n. Notice that

$$\mu_N(C_0) = \frac{|A \cap [-N, N]|}{2N+1}$$

and so $\mu_N(C_0)$ is the density of A in in $\{-N, ..., N\}$. As A has positive upper density d, then we may pick a subsequence $N_k \to \infty$ so that $\mu_{N_k}(C_0) \to d > 0$.

By identifying $C(X)^*$ with the space of finite Borel measures on X and applying Alaoglu's theorem, we know that the set of probability measures is compact with respect to the weak* topology. Moreover, since X is a compact metric space, then $C(X)$ is separable; therefore the weak* topology on $C(X)^*$ is metrizable. So the set of Borel probability measures on Y is sequentially compact.

Therefore choose a weak* limit point μ of $\{\mu_{N_k}\}$. Then $\mu(C_0) > 0$ as $\lim_k \mu_{N_k}(C_0) > 0$. Furthermore

$$\mu_N \circ T^{-1} - \mu_N = \frac{1}{2N+1}(\delta_{a_{N+1}} - \delta_{a_{-N}}).$$

Hence if f is a continuous function on X (and so bounded), then

$$\int f \;\; d(\mu_N \circ T^{-1} - \mu_N) = \frac{1}{2N+1}(f(a_{N+1}) - f(a_{-N})) \to 0$$

as $N \to \infty$. Thus μ, being a weak* limit point, is T-invariant.
So $(X, \mathcal{B}, \mu, T)$ is a measure preserving system with $\mu(C_0) > 0$, and so theorem 0.5 implies that $\mu(\bigcap_{j=1}^{k-1} T^{-jb} C_0) > 0$ for all k and some b depending on k. □

References

[Fur81a] H. Furstenberg. *Recurrence in Ergodic Theory and Combinatorial Number Theory.* Princeton, NJ: Princeton University Press, 1981.

[Fur81b] Harry Furstenberg. "Poincaré recurrence and number theory". In: *Bulletin of the American Mathematical society* 5.3 (1981), pp. 211–234.

[FKO82] Hillel Furstenberg, Yitzhak Katznelson, and Donald Ornstein. "The ergodic theoretical proof of Szemerédis theorem". In: *Bulletin of the American Mathematical Society* 7.3 (1982), pp. 527–552.

[EW11] Manfred Einsiedler and Thomas Ward. *Ergodic Theory. With a view towards number theory.* New York, NY: Springer-Verlag, 2011.