Entropy, Planck, and Zeta

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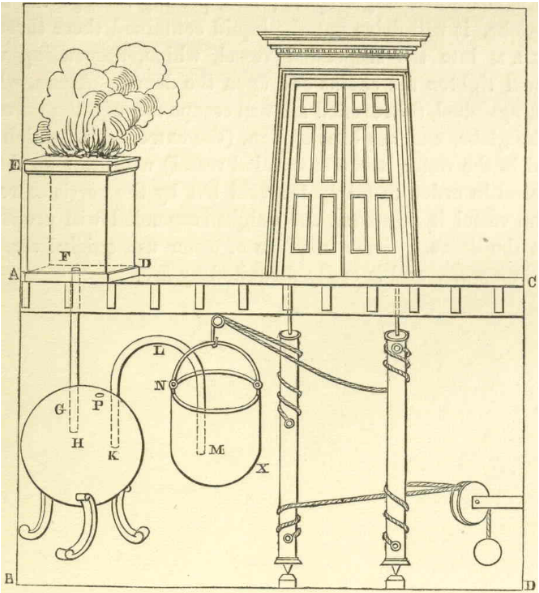
$$m_p = \int_0^\infty \frac{8\pi h}{c^3} \frac{\nu^{3+p}}{e^{\frac{h\nu}{kT}} - 1} \, d\nu = \frac{8\pi h}{c^3} \left(\frac{kT}{h} \right)^{4+p} \Gamma(4+p) \, \zeta(4+p)$$

- Hero's pneumatic doors.
- Boyle's Law.
- Bernoulli's pressure.
- Charles's & Gay-Lussac's Laws.
- First Law(s).
- Entropy:
  - ▷ Carnot's engine.
  - ▷ Stefan's law.
  - ▷ Wien's displacement law.
  - ▷ Planck's black body equation.
- Riemann's $\zeta(z)$ in pyrometry.



Q1: How much wood?

Q2: In Summer? Winter?

Q3: Are air & water best?

Q4: First thermometer?

Q5: What *is* temperature?

Carnot answers Q1–Q3 in early 1800's.

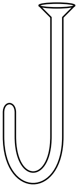
Torricelli and others answer Q4 in 1600's.

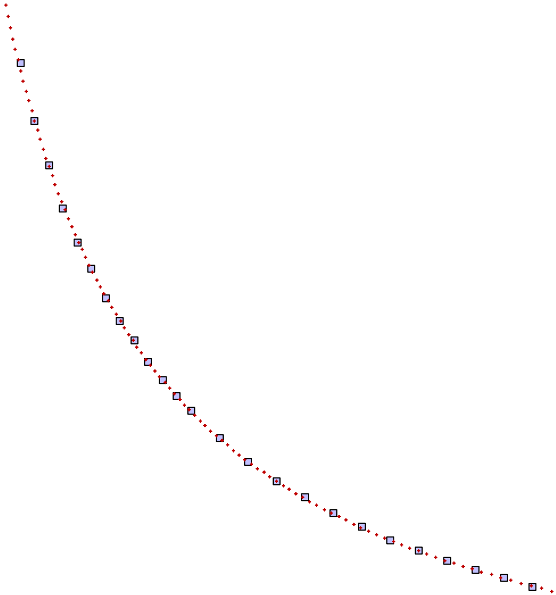
Boyle shows that for air at constant temperature,

$$PV = k.$$

So, $PV = F(T)$.

No thermometers required!





Pressure from "corpuscle" collisions:

$$P = \frac{\text{Time average normal force}}{\text{area}}$$
$$= \frac{1}{A}\frac{1}{\tau}\int_0^\tau \mathsf{F}_n \, dt = \frac{1}{A}\frac{1}{\tau}\sum_i \Delta \mathsf{p}_i^T \mathsf{n},$$

sum over all collisions in $0 \le t \le \tau$.

On the right wall, only $m\dot{x}$ changes in $\Delta \mathsf{p}_i$ because F_n points in the x-direction. So,

$$\Delta m_i \dot{y}_i = \Delta m_i \dot{z}_i = 0.$$

If energy is conserved, then

$$\frac{1}{2} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)^+ = \frac{1}{2} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)^- .$$

So $\dot{x}_i^+ = -\dot{x}_i^-$ & $\Delta \mathsf{p}_i^T \mathsf{n} = 2 m_i |\dot{x}_i|$.

Cor: angle of incidence = angle of reflection.

hits from a single corpuscle for $t \in (0, \tau)$ is

$$\# \text{ hits} = \tau \frac{|\dot{x}|}{2\ell},$$

where ℓ is the box's length. So

$$P_{\text{right}} = \frac{1}{A}\frac{1}{\tau}\sum_i 2 m_i |\dot{x}_i|$$

$$= \frac{1}{A\tau}\sum_j \# \text{ hits} \cdot 2 m_j |\dot{x}_j| = \frac{2}{V}\sum_j \frac{1}{2} m_j \dot{x}_j^2,$$

where sums on j are over all corpuscles.

Average over all the walls:

$$P = \frac{2}{3}\frac{\sum_j \frac{1}{2} m_j |\mathsf{v}_j|^2}{V}. \qquad (P_B)$$

In English: Bernoulli's

$$P = \frac{2}{3} \frac{\sum_j \frac{1}{2} m_j \left|\mathsf{v}_j\right|^2}{V}.$$

means the Pressure is $\frac{2}{3}$ of the kinetic energy per unit volume.

A similar result holds for radiated energy, where $P = \frac{1}{3}$ of the energy density $u = \frac{U}{V}$.

By Boyle's Law, $PV = F(T)$, so temperature is related to energy.

Bernoulli's results were largely ignored.
Nobody believed corpuscles whiz about.

All of our examples will have this form:

$$U = \text{energy}$$
$$V = \text{volume}$$
$$P = \text{pressure} = \beta \frac{U}{V} = \beta u.$$

Bernoulli's computation for gasses puts

$$P = \frac{2}{3} u.$$

Electromagnetism's computation puts

$$P = \frac{1}{3} u.$$

Jacques Charles shows

$$V = k(T + b)$$

at constant pressure.

Charles was an avid balloonist.

Result makes Hero's machine a thermometer (thermoscope).

Joseph-Louis Gay-Lussac shows

$$P = k(T + b)$$

at constant volume.

Gay-Lussac credits Charles's (unpublished) Law.

TIL (Thing I've Learned): In this era, "y is proportional to x" means

$$y = k(x + b)$$

and "y is directly proportional to x" means

$$y = kx.$$

For T in degrees Celsius, the value

$$b = 273$$

is well-known, but Kelvin won't propose his scale until 1848.

Julius Robert Mayer and James Joule each posit that

- Heat is a form of energy, and that
- Energy is conserved.

"Internal energy" U is heat ΔQ added minus work $\int P\, dV$ done. The dynamical system is

$$\frac{d}{dt}U = \frac{d}{dt}Q - P\frac{d}{dt}V$$

Heat Q is a "non-homogeneous" or "driving force".

It is often a "boundary term" (fire on an altar), a "body term" (gasoline engine), or frictional.

The important point is that Q is not a function of "state" variables P, V, T, U. We may know $\dot{Q}$ from reading a power meter, but $\dot{Q}$ is not "exact" — you can't find ΔQ knowing only the start and end values of P, V, T, U. ΔQ is path dependent, just as is work.

Engines run in cycles: heat drives the engine, work is done, heat is exhausted, the system recovers — and repeats.

The notation $\dot{X}$ means $\frac{d}{dt}X$ (as usual).

Suppose that $PV = \beta U = \alpha T$.
The First Law, $\dot{U} = \dot{Q} - P\dot{V}$, becomes

$$\begin{aligned}\frac{d}{dt}Q &= \frac{d}{dt}U + P\frac{d}{dt}V \\ &= \frac{\alpha}{\beta}\frac{d}{dt}T + \frac{\alpha T}{V}\frac{d}{dt}V\end{aligned}$$

What to do? $\frac{1}{T}$ is an integration factor:

$$\begin{aligned}\frac{\frac{d}{dt}Q}{T} &= \frac{\alpha}{\beta}\frac{\frac{d}{dt}T}{T} + \alpha\frac{\frac{d}{dt}V}{V} \\ &= \frac{d}{dt}\underbrace{\frac{\alpha}{\beta}\log\left(TV^{\beta}\right)}_{S=S(T,V)}.\end{aligned}$$

S is the entropy of the system (to within an additive constant). It is a state variable.

In the simplest problem,

$$\frac{d}{dt}Q = T\frac{d}{dt}S.$$

The First Law says this simple result is universally true and that

$$\frac{d}{dt}U = T\frac{d}{dt}S - P\frac{d}{dt}V. \qquad (L_1)$$

Note: $\dot{Q} = T\dot{S}$ says $\frac{1}{T}$ is always an integration factor for $\dot{Q}$, no matter what equation of state holds between P, V, T, and U. Hmmm!

Note: In adiabatic systems, $\dot{S} = 0$ (so $\dot{Q} = 0$).
In isothermal systems, $\dot{T} = 0$.
In these two (very important) cases, I can explicitly compute ΔQ without entropy.

So what *is* entropy? Whatever it takes to make $\dot{Q} = T\dot{S}$ so that the first law is

$$\frac{d}{dt}U = T\frac{d}{dt}S - P\frac{d}{dt}V. \qquad (L_1)$$

Ok, fine. I understand P and V and maybe T. So what is U? Whatever it takes to make (L_1) true.

TIL: I believe thermodynamics consists largely of finding a pair of functions S and $U(V, S)$ for which (L_1) models a given system.

Once found, $\dot{Q} = T\dot{S}$.

Planck wants S for radiating systems.

Entropy makes every problem look (at least a little) like an ideal gas problem.

The first law,

$$\frac{d}{dt}U = T\frac{d}{dt}S - P\frac{d}{dt}V.$$

plus the chain rule means the "natural variables" of U are S and V, and that

$$\begin{aligned} \frac{\partial}{\partial S}U(S,V) &= \quad T \\ \frac{\partial}{\partial V}U(S,V) &= -P. \end{aligned}$$

The top line defines temperature (if you are optimistic (gullible?)).
The internal energy U is especially useful when

1) S is constant (adiabatic processes), or
2) V is constant (isochoric processes)

The product rule says T.F.A.E.:

$$\frac{d}{dt}U = T\frac{d}{dt}S - P\frac{d}{dt}V$$

$$\frac{d}{dt}\underbrace{(U+PV)}_{H} = T\frac{d}{dt}S + V\frac{d}{dt}P$$

$$\frac{d}{dt}\underbrace{(U-TS)}_{A,F} = -S\frac{d}{dt}T - P\frac{d}{dt}V$$

$$\frac{d}{dt}\underbrace{(U-TS+PV)}_{G} = -S\frac{d}{dt}T + V\frac{d}{dt}P$$

The names and "natural" variables are

$$H = H(S,P) = \text{enthalpy}$$

$$A = A(T,V) = \text{Helmholtz energy}$$

$$G = G(T,P) = \text{Gibbs free energy}$$

Thermodynamicists call this product rule the Legendre Transform.

They are much better at mnemonics: "The Amiable (Friendly) Vagabond Usually Shares His Precious Gifts":

$$\begin{matrix} T & A & V \\ G & & U \\ P & H & S \end{matrix}$$

Then

$$A = A(T,V)$$

$$G = G(P,T) \quad U = U(V,S)$$

$$H = H(S,P)$$

Theorem: If $PV = \alpha T$, then $U = U(T)$.

Proof: From $\dot{U} = T\dot{S} - P\dot{V}$,

$$\dot{S} = \frac{\dot{U} + P\dot{V}}{T} = \frac{\dot{U}}{T} + \frac{\alpha \dot{V}}{V}$$

Think: $U = U(T, V)$ and $S = S(T, V)$. Then

$$\dot{S} = \frac{\partial U}{\partial T}\frac{1}{T}\dot{T} + \left(\frac{\partial U}{\partial V}\frac{1}{T} + \frac{\alpha}{V}\right)\dot{V}$$

so

$$\begin{cases} \dfrac{\partial S(T,V)}{\partial T} = \dfrac{\partial U}{\partial T}\dfrac{1}{T} & \text{and} \\ \dfrac{\partial S(T,V)}{\partial V} = \dfrac{\partial U}{\partial V}\dfrac{1}{T} + \dfrac{\alpha}{V} \end{cases}$$

Equality of mixed partials means

$$\frac{\partial^2 U}{\partial V \partial T}\frac{1}{T} = \frac{\partial^2 U}{\partial T \partial V}\frac{1}{T} - \frac{\partial U}{\partial V}\frac{1}{T^2}$$

Conclusion:

$$\frac{\partial U(T,V)}{\partial V} = 0.$$

QED.

Note that entropy S appears in the proof, but not in the statement or the conclusion.

Since $\dot{Q} = T\dot{S}$,

$$\Delta S = \int \frac{\dot{Q}}{T}\,dt = \int \frac{\dot{Q}^+}{T} - \frac{\dot{Q}^-}{T}\,dt$$

$$\geq \int \frac{\dot{Q}^+}{T_{\max}} - \frac{\dot{Q}^-}{T_{\min}}\,dt = \frac{Q_{\text{in}}}{T_{\max}} - \frac{Q_{\text{out}}}{T_{\min}}.$$

If the path is a cycle, then $\Delta S = 0$ and

$$\frac{Q_{\text{out}}}{T_{\min}} \geq \frac{Q_{\text{in}}}{T_{\max}}.$$

Consequently,

$$\Delta Q = Q_{\text{in}} - Q_{\text{out}} \leq Q_{\text{in}}\left(1 - \frac{T_{\min}}{T_{\max}}\right)$$

Since ΔQ is the work done,

$$\text{Work done} \leq Q_{\text{in}}\left(1 - \frac{T_{\min}}{T_{\max}}\right),$$

answering "Q1: How much wood?".

$\left(1 - \frac{T_{\min}}{T_{\max}}\right)$ is the engine's efficiency.

The Carnot engine realizes the most efficient engine possible:

- $T = T_{\max}$ whenever $\dot{Q}^+ \neq 0$,
- $T = T_{\min}$ whenever $\dot{Q}^- \neq 0$, and
- $\dot{Q} = 0$ whenever $\dot{T} \neq 0$.

The top two processes are isothermal ($\dot{T} = 0$); the bottom is adiabatic ($\dot{Q} = 0$).

(L_1) says entropy S exists and $T\dot{S} = \dot{Q}$.

The Second Law of thermodynamics says no heat engine is more efficient than Carnot's.

The Second Law is used to prove perpetual motion and "free work" do not exist, and that the "arrow of time" works against man.

In the 1800s and early 1900s, the Second Law was the focus of intense study. Many thought it more fundamental than atomic theory. Planck didn't give a whit about atoms — he pursued more fundamental theories.

See C. P. Snow's Rede Lecture "Two Cultures", then give Flanders & Swann's "First and Second Laws" a listen.

Carnot deduced all his results from physical arguments without the benefit of entropy.

Note that he has answered Q1–Q3 from Slide 3.

Joseph Stefan conjectured that radiated energy and temperature satisfy

$$\frac{U}{V} = u \propto T^4.$$

Planck's approach uses entropy from (L_1):

$$\dot{S} = \frac{\dot{U} + P\dot{V}}{T} = \frac{\dot{u}V + (u+P)\dot{V}}{T}.$$

For radiation, $P = \frac{1}{3}u$, so

$$\dot{S} = \frac{V\dot{u} + \frac{4}{3}u\dot{V}}{T}.$$

Choose independent variables T and V. Since $u = u(T)$,

$$\dot{S}(T,V) = \frac{Vu'(T)\dot{T} + \frac{4}{3}u(T)\dot{V}}{T}.$$

The Chain Rule says

$$\frac{\partial S(T,V)}{\partial T} = \frac{Vu'(T)}{T}. \quad \text{and} \quad \frac{\partial S(T,V)}{\partial V} = \frac{\frac{4}{3}u(T)}{T}$$

Equality of mixed partials implies

$$\frac{\partial^2 S(T,V)}{\partial V \partial T} = \frac{u'(T)}{T}$$
$$= \frac{\partial^2 S(T,V)}{\partial T \partial V} = \frac{4}{3}\frac{u'(T)}{T} - \frac{\frac{4}{3}u(T)}{T^2}.$$

The energy density $u(T)$ therefore satisfies

$$\frac{1}{3}u'(T) = \frac{\frac{4}{3}u(T)}{T}.$$

From

$$\frac{1}{3}u'(T) = \frac{\frac{4}{3}u(T)}{T},$$

we conclude

$$\log(u(T)) = 4\log(T) + C$$

or

$$u(T) \propto T^4.$$

Wait[3]: Who said $u(T,V) = u(T)$? Planck says so before the theorem, but all others I've seen impose that condition in the middle of the proof. That's OK: Our job is to find u (and S) for which (L_1) models reality. Check!

"Chunks" of energy with density u whiz about in all directions at the speed of light. If one of those chunks falls on a solar panel, the power per unit area is

$$\text{ppua} = c \cdot u = \sigma T^4.$$

The Stefan-Boltzmann constant is

$$\sigma \approx 5.67 \times 10^{-8} \ \frac{\text{W}}{\text{m}^2\,\text{K}^4}.$$

Stefan's law includes energy from all frequencies of radiation. Wien wanted the energy "density" $\mu(\nu, T)$ where

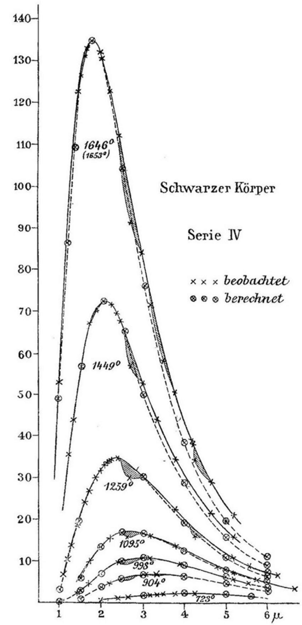
$$u(T) = \int_0^\infty \mu(\nu, T)\, d\nu.$$

To tease out the dependence on ν, Wien imagined an oven expanding adiabatically ($\dot{Q} = 0$), and asked how the Doppler shift changes ν.

$$\frac{d}{dt}U = T\frac{d}{dt}S - P\frac{d}{dt}V$$
$$= 0 - \frac{1}{3}\frac{U}{V}\frac{d}{dt}V.$$

so

$$UV^{\frac{1}{3}} = uV^{\frac{4}{3}} = \text{const. when } \dot{S} = 0.$$



Stefan's law says

$$u(T) \propto T^4$$

so

$$uV^{\frac{4}{3}} \propto T^4 V^{\frac{4}{3}} = \text{const.}$$

With $V \propto R^3$,

$$TR = \text{const.}$$

Pick a wavelength λ. Maxwell's equations imply

$$\lambda \propto R,$$

so

$$T\lambda = \text{const.} \qquad \text{or} \qquad \frac{\nu}{T} = \text{const.}$$

as the oven expands.

In other words, the graphs dilate with T:

$$\lambda_1 T_1 = \lambda_2 T_2 \qquad \text{or} \qquad \frac{\nu_1}{T_1} = \frac{\nu_2}{T_2}.$$

This is Wien's Displacement Law.

The obvious (and useful) point to watch while dilating is the maximum of the graph. If $\lambda = \lambda_{\max}$, then

$$\lambda_{\max} T \approx 2.9 \times 10^{-3} \text{m K}.$$

Wait^3: Who said the *vertical* dilation was uniform? If the vertical dilation depends on λ, then the maximum shifts.

Planck says so. After a detailed examination of how the Doppler shift shifts frequencies as the oven expands, he concludes that the density $\mu(\nu, T)$ satisfies

$$\frac{\partial}{\partial V}\left(\mu(\nu, V)V\right) = \frac{\nu}{3}\frac{\partial \mu}{\partial \nu}.$$

(Recall that adiabats satisfy $V^{\frac{1}{3}}T = \text{const.}$, so T and V are interchangeable.)
The general solution is

$$\mu(\nu, V) = \frac{\phi(V\nu^3)}{V} \qquad \text{or} \qquad \mu(\nu, T) = T^3\varphi\left(\frac{\nu}{T}\right).$$

The vertical dilation factor is T^3 — the same factor for every ν.

Furthermore, Stefan's Law follows from Wien's analysis:

$$\begin{aligned} u(T) = \int_0^\infty \mu(\nu, T)\, d\nu &= \int_0^\infty T^3\varphi\left(\frac{\nu}{T}\right) d\nu \\ &= T^4 \int_0^\infty \varphi(x)\, dx, \end{aligned}$$

and the last integral is independent of T.

"We pause here for a moment to marvel at Wien's clever use of thermodynamics to provide insights into the nature of black-body spectrum." (Manoj K. Harbola, "The Genesis of Quanta: 1890-1910" p. 148)

Q: How hot is the sun?

The sun is yellow-ish, say $\approx 580 \times 10^{-9}$m.

Wien predicts $\lambda_{\max} T \approx 2.9 \times 10^{-3}$m K, so

$$T_{\text{sun}} \approx \frac{2.9 \times 10^{-3}}{580 \times 10^{-9}} = 5000\ K.$$

The sun's surface temperature is actually about 5800 K.

Is the sun a black body?

Q: How far is the earth from the sun?

A: Stefan's law says the sun radiates power

$$p_{\text{sun}} = \sigma 5800^4$$

from its surface. Some of that power hits the earth at about 1.4 kW/m^2. If the earth is D sun-radii away from the sun, then the sun radiates at rate

$$1.4 \cdot D^2 = \sigma 5800^4.$$

So

$$D \approx \sqrt{\frac{5.67 \times 10^{-8} \cdot 5800^4}{1400}} \approx 215.$$

D is actually about

$$\frac{93 \times 10^6}{433,000} \approx 215.$$

Planck balanced the radiation energy density $\mu(\nu, T)$ with the energy E of radiation of an oscillator in the wall of the oven:

$$\mu(\nu, T) = \frac{8\pi\nu^2}{c^3} E(\nu, T).$$

Naturally, he wants the entropy $S(E)$ associated with the oscillator. The First Law will be

$$\dot{E} = T\dot{S} - P\dot{V}$$

so that

$$\dot{S} = \frac{1}{T}\dot{E} + \frac{P}{T}\dot{V}.$$

The chain rule says

$$\frac{\partial S(E, V)}{\partial E} = \frac{1}{T}.$$

Planck's firm belief in the importance of the Second Law drives him to look for thermal equilibrium when S is maximized.

At the time, there were several competing models of μ, leading to several competing models for E. Planck analyzes entropy for them as follows:

Wien's model proposes

$$E(\nu, T) \propto \nu e^{-b\nu/T} \qquad (E_W)$$

Since

$$\frac{\partial S}{\partial E} = \frac{1}{T} \propto \frac{-1}{b\nu} \log\left(\frac{E}{a\nu}\right),$$

Planck integrated and *defined*

$$S = \frac{-a}{b}\left(\frac{E}{a\nu}\log\left(\frac{E}{a\nu}\right) - \frac{E}{a\nu}\right).$$

The Second Law is *very* important to Planck, so he computes

$$\frac{\partial^2 S}{\partial E^2} = \frac{-1}{b\nu E} < 0.$$

Wien's model holds for small T and large ν, so small E.

The Rayleigh-Jeans model proposes

$$E(\nu, T) = kT. \qquad (E_{RJ})$$

Integrate

$$\frac{\partial S}{\partial E} = \frac{1}{T} = \frac{k}{E},$$

and *define*

$$S = k \log(E).$$

Compute

$$\frac{\partial^2 S}{\partial E^2} = \frac{-1}{E^2} < 0.$$

(E_{RJ}) holds for large T and small ν, so large E.

What to do?

$$\frac{\partial^2 S}{\partial E^2} \propto \frac{-1}{E}$$

looks good for small E, and

$$\frac{\partial^2 S}{\partial E^2} \propto \frac{-1}{E^2}$$

looks good for large E.

How about

$$\frac{\partial^2 S}{\partial E^2} = \frac{-a}{E(E+b)}?$$

Use partial fractions to integrate once:

$$\frac{\partial S}{\partial E} = \int \frac{\frac{a}{b}}{E+b} - \frac{\frac{a}{b}}{E}\, dE = \frac{a}{b} \log\left(\frac{E+b}{E}\right).$$

Since $\frac{\partial S}{\partial E} = \frac{1}{T}$,

$$\frac{E+b}{E} = e^{\frac{b}{aT}},$$

so

$$E = \frac{b}{e^{\frac{b}{aT}} - 1},$$

so

$$\mu(\nu, T) = \frac{8\pi\nu^2}{c^3} E(\nu, T) = \frac{8\pi\nu^2}{c^3} \frac{b}{e^{\frac{b}{aT}} - 1}.$$

Match some constants to find that $b = h\nu$ (where Planck didn't know h yet) and $a = k$.

I don't know how you'd "interpolate" between the Wien and Rayleigh-Jeans models except by interpolating between the two $\frac{\partial^2 S}{\partial E^2}$s.

Think of the black body density

$$\mu(\nu, T) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{\frac{h\nu}{kT}} - 1} = \frac{8\pi h}{c^3}\frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1}$$

as a density. The p^{th} moment of that density is

$$m_p = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^{3+p}}{e^{J\nu} - 1}\,d\nu$$

where $J = h\nu/k$. Expand the geometric series:

$$\begin{aligned}\int_0^\infty \frac{\nu^{3+p}}{e^{J\nu} - 1}\,d\nu &= \int_0^\infty \nu^{3+p}\frac{e^{-J\nu}}{1 - e^{-J\nu}}\,d\nu \\ &= \int_0^\infty \nu^{3+p}\left(e^{-J\nu} + e^{-2J\nu} + e^{-3J\nu} + \cdots\right)\,d\nu \\ &= \sum_{\ell=1}^\infty \int_0^\infty \nu^{3+p}e^{-\ell J\nu}\,d\nu.\end{aligned}$$

An individual term looks like

$$\begin{aligned}\int_0^\infty \nu^{3+p}e^{-\ell J\nu}\,d\nu &= \frac{1}{(\ell J)^{4+p}}\int_0^\infty x^{4+p-1}e^{-x}\,dx \\ &= \frac{1}{(\ell J)^{4+p}}\Gamma(4+p)\end{aligned}$$

($\Gamma(4+p) = (3+p)!$ for integer $p > -4$. You can explicitly compute these values by integrating by parts.)

Sum over the individual terms:

$$\begin{aligned}\int_0^\infty \frac{\nu^{3+p}}{e^{J\nu} - 1}\,d\nu &= \Gamma(4+p)\sum_{\ell=1}^\infty \frac{1}{(\ell J)^{4+p}} \\ &= \frac{\Gamma(4+p)}{J^{4+p}}\sum_{\ell=1}^\infty \frac{1}{\ell^{4+p}} = \frac{\Gamma(4+p)}{J^{4+p}}\zeta(4+p).\end{aligned}$$

For example, $p = 0$ and $J = \frac{h}{kT}$ says

$$m_0 = \int_0^\infty \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1}\, d\nu = \frac{3!k^4}{h^4}\zeta(4)T^4.$$

This is Stefan's law:

$$\text{radiated energy} \propto T^4.$$

The black body density has the generating function of the Bernoulli numbers in it:

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Bennet Woodcroft trans. & ed., *The Pneumatics of Hero of Alexandria*, Taylor, Walton and Maberly, London, 1851, Invention "37. Temple Doors Opened by Fire on an Altar", p. 57. Freely available as https://tile.loc.gov/storage-services/service/rbc/rbc0001/2009/2009gen41532/2009gen41532.pdf

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O. Lummer and E. Pringsheim's graph from https://www.researchgate.net/figure/Black-body-radiation-at-various-temperatures-as-measured-by-E-Lummer-and-E-Pringsheim_fig4_339664052

Manoj K Harbola, "The Genesis of Quanta: 1890-1910", Resonance, February 2008, 13(2):134-171. DOI:10.1007/s12045-008-0029-6.

C. P. Snow's Rede Lecture (1959) "Two Cultures" is available at https://apps.weber.edu/wsuimages/michaelwutz/6510.Trio/Rede-lecture-2-cultures.pdf.

Michael Flanders and Donald Swann, "First and Second Laws" is worth a listen.

Lawrence C. Evans, "Entropy and PDE", https://math.berkeley.edu/ evans/entropy.and.PDE.pdf

Max Planck, Morton Masius, trans., *The Theory of Heat Radiation*, P. Blakiston's Son & Co., Philadelphia, 1914. Available as Project Gutenberg eBook #40030, https://www.gutenberg.org/files/40030/40030-pdf.pdf.