An Extreme Value Type Theorem for Convex Functions on Reflexive Banach Spaces

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May 23, 2024

Abstract

We will review the interplay between weak topologies, convexity of sets and functions, and lower semicontinuity and will apply the results to optimization and the existence of minimizers in an infinite-dimensional setting.

1 Topologies and Continuous Functions [3]

Definition 1. A TOPOLOGICAL SPACE is a pair (X, τ) where X is a non-empty set and τ is a collection of subsets of X referred to as a TOPOLOGY on X which satisfies:

(i) $\emptyset, X \in \tau$;

(ii) if $\{U_\alpha\}_{\alpha \in I} \subseteq \tau$, then $\cup_\alpha U_\alpha \in \tau$; and

(iii) if $\{U_1, \ldots, U_n\} \subseteq \tau$ for some $n \in \mathbb{N}$, then $\cap_i U_i \in \tau$.

A subset U of X is an OPEN SET if $U \in \tau$, and a subset K of X is a CLOSED SET if $K^c \in \tau$. If the topology is understood or does not need to be specified, we will refer to X as a topological space.

Definition 2. Given a topological space (X, τ), a subset $\mathcal{B} \subseteq \tau$ is a BASIS FOR τ if for every $U \in \tau$ there is a subset $\mathcal{B}' \subseteq \mathcal{B}$ such that $U = \cup_{V \in \mathcal{B}'} V$.

The following lemma provides an alternate characterization of a basis for a topology

Lemma 3. *Let X be a set, and let $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfy:*

(a) for any $U, V \in \mathcal{B}$ and $x \in U \cap V$ there is a $W \in \mathcal{B}$ such that $x \in W \subseteq U \cap V$; and

(b) $X = \cup_{U \in \mathcal{B}} U$.

Then there is a topology τ on X such that $\mathcal{B}$ is a basis for (X, τ).

Proof. See [3, Theorem 1.18]. □

The main example of a topological space for this presentation will be a Banach space (Definition 8). The norm-induced metric provides the norm topology where open balls comprise a basis for the topology.

Definition 4. Let X, Y be topological spaces. A function $f : X \to Y$ is CONTINUOUS AT $x \in X$ if for every open $V \subseteq Y$ containing $f(x)$ there is an open $U \subseteq X$ containing x such that $f(U) \subseteq V$. A function $f : X \to Y$ is CONTINUOUS if it is continuous at every $x \in X$.

Proposition 5. *A function $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is an open subset of X for every open subset V of Y.*

Proof. Let V be open in Y and choose $x \in f^{-1}(V)$. Since V is open, $f(x) \in V$, and f is continuous, there is an open set U containing x such that $f(U) \subseteq V$. We now have an open set U such that $x \in U \subseteq f^{-1}(V)$ and since x was arbitrarily chosen in $f^{-1}(V)$, we conclude $f^{-1}(V)$ is open.

Conversely, let $x \in X$ and suppose V is open in Y and $f(x) \in V$. In that case, $x \in f^{-1}(V)$ which is open in X, so there is an open $U \subseteq X$ such that $x \in U \subseteq f^{-1}(V)$. It follows that $f(U) \subseteq V$, thus f is continuous. □

The following result applies to a more general setting where the domain space X is only a set without a topology and the functions are therefore only functions and not continuous (since continuity has no meaning without a topology on X). However, restricting to a subset of a given topology on X is of interest for this presentation.

Proposition 6. *Let $\mathcal{F}$ be a family of continuous functions between topological spaces (X, τ_X) and (Y, τ_Y). The collection $\mathcal{B}$ defined by*

$$\mathcal{B} = \{\cap_{i=1}^n f_i^{-1}(V_i) : f_i \in \mathcal{F},\, V_i \in \tau_Y,\, n \in \mathbb{N}\}$$

forms a basis for a topology $\tau_{\mathcal{F}}$ on X which is a subset of τ_X and is the weakest topology on X such that all of the functions in $\mathcal{F}$ are continuous.

Proof. We will first apply Lemma 3 to show that $\mathcal{B}$ is a basis for a topology on X. Let $U_1, U_2 \in \mathcal{B}$ have non-empty intersection and let $x \in U_1 \cap U_2$. Then

$$U_1 = \cap_{i=1}^m f_i^{-1}(V_i) \text{ for some } m \in \mathbb{N},\, f_1, \ldots, f_m \in \mathcal{F},\, V_1, \ldots, V_m \in \tau_Y$$

and

$$U_2 = \cap_{i=1}^n g_i^{-1}(W_i) \text{ for some } n \in \mathbb{N},\, g_1, \ldots, g_n \in \mathcal{F},\, W_1, \ldots, W_n \in \tau_Y$$

and $U = U_1 \cap U_2 \in \mathcal{B}$ satisfies $x \in U \subseteq U_1 \cap U_2$.

Next, for any $x \in X$ pick a function $f \in \mathcal{F}$ and then $Y \in \tau_Y$ and $x \in U := f^{-1}(Y) \in \mathcal{B}$, thus $X = \cup_{U \in \mathcal{B}} U$. Lemma 3 gives that $\mathcal{B}$ is the basis for a topology on X.

Next, every element of $\mathcal{B}$ belongs to τ_X since every function in $\mathcal{F}$ is continuous relative to τ_X, so $\mathcal{B} \subseteq \tau_{\mathcal{F}} \subseteq \tau_X$. And for any $f \in \mathcal{F}$ and any open $U \in \tau_Y$, $f^{-1}(U) \in \mathcal{B} \subseteq \tau_{\mathcal{F}}$ thus every function in $\mathcal{F}$ is continuous relative to $\tau_{\mathcal{F}}$.

Lastly, if τ'_X is another topology for X such that every function in $\mathcal{F}$ is continuous, then $\mathcal{B} \subseteq \tau'_X$ thus $\tau_{\mathcal{F}} \subseteq \tau'_X$. □

In the above case, $\tau_{\mathcal{F}} \subseteq \tau_X$ so any open set in $\tau_{\mathcal{F}}$ is open in τ_X, and equivalently any closed set in $\tau_{\mathcal{F}}$ is closed in τ_X. We refer to $\tau_{\mathcal{F}}$ as the weak topology generated by $\mathcal{F}$.

Proposition 7. *Let $\mathcal{F}$ be a family of continuous functions between topological spaces X and Y, and equip X with the weak topology generated by $\mathcal{F}$. A net $\{x_\alpha\}_{\alpha \in A} \subseteq X$ converges weakly (i.e., in $\tau_{\mathcal{F}}$) to a point $\bar{x} \in X$ if and only if $f(x_\alpha) \to f(\bar{x})$ in Y for every $f \in \mathcal{F}$.*

Proof. Let $f \in \mathcal{F}$, and let V be an open set in Y such that $f(\bar{x}) \in V$. Now $f^{-1}(V)$ is a weakly open set containing $\bar{x}$ and since $\{x_\alpha\}$ converges weakly to $\bar{x}$, there is an $\bar{\alpha} \in A$ such that $x_\alpha \in f^{-1}(V)$ and therefore $f(x_\alpha) \in V$ whenever $\bar{\alpha} \leq \alpha$. That is, $\{f(x_\alpha)\}$ is eventually in V so we conclude $f(x_\alpha) \to f(\bar{x})$ since V was an arbitrary open set containing $f(\bar{x})$.

Conversely, let U be a weakly open set containing $\bar{x}$. In that case, by Proposition 6 there are functions $f_1, \ldots, f_n \in \mathcal{F}$ and open sets $V_1, \ldots, V_n \subseteq Y$ such that $\bar{x} \in \cap_{i=1}^n f_i^{-1}(V_i) \subseteq U$, and thus $f_i(\bar{x}) \in V_i$ for every i. Since each $f_i(x_\alpha) \to f_i(\bar{x})$, there is an $\alpha_i \in A$ such that $f_i(x_\alpha) \in V_i$ whenever $\alpha_i \leq \alpha$. Since A is a directed set, it follows that there is an $\bar{\alpha} \in A$ such that $\alpha_i \leq \bar{\alpha}$ for $i = 1, \ldots, n$, and then $x_\alpha \in \cap_{i=1}^n f_i^{-1}(V_i) \subseteq U$ whenever $\bar{\alpha} \leq \alpha$ so $\{x_\alpha\}$ is eventually in U. Since U was an arbitrary weakly open set containing $\bar{x}$, we conclude $x_\alpha \to \bar{x}$ weakly. □

2 Reflexive Banach Spaces [1], [2]

Definition 8. A real BANACH SPACE is a normed vector space with scalar field $\mathbb{R}$ which is also a complete metric space with regard to the metric induced by the norm.

That is, if X is a Banach space, all Cauchy sequences in X converge to a limit in X. Namely, for all $\epsilon > 0$, if $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ satisfies $d(x_n, x_m) = \|x_n - x_m\| < \epsilon$ whenever $m, n > N$ for some $N \in \mathbb{N}$, there is a point $\bar{x} \in X$ such that $x_n \to \bar{x}$.

Definition 9. Given a normed vector space X, the DUAL OF X (or X^*) is the set of continuous linear functionals from X to $\mathbb{R}$. For $x \in X$ and $x^* \in X^*$, we use the usual notation $\langle x^*, x\rangle = x^*(x)$.

Note that X^* equipped with the operator norm is a normed vector space, and it is always a Banach space whether or not X is.

Proposition 10. *Let X and Y be normed vector spaces where Y is a Banach space. The set $\mathcal{B}(X, Y)$ of all continuous linear functions from X to Y is a Banach space.*

Proof. Let $\{T_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}(X, Y)$, let $x \in X \setminus \{0\}$, and let $\epsilon > 0$. Since $\{T_n\}$ is a Cauchy sequence, there is a number N such that

$$\|T_n - T_m\| \leq \frac{\epsilon}{\|x\|} \text{ whenever } m, n > N.$$

Now $\{T_n x\}$ is a sequence in Y, and

$$\|T_n x - T_m x\| = \|(T_n - T_m)x\| \leq \|T_n - T_m\|\|x\| < \epsilon \text{ whenever } n, m > N,$$

so $\{T_n x\}$ is a Cauchy sequence in Y. Since Y is a Banach space, there is a vector $Tx \in Y$ such that $T_n x \to Tx$. We must show that the resulting operator T belongs to $\mathcal{B}(X, Y)$ and that $T_n \to T$ in $\mathcal{B}(X, Y)$.
To see that T is linear, let $x, y \in X$, and then

$$\begin{aligned}
T(x+y) &= \lim T_n(x+y) && \text{(definition of } T\text{)}\\
&= \lim(T_n x + T_n y) && \text{(each } T_n \text{ is linear)}\\
&= \lim T_n x + \lim T_n y && \text{(continuity of addition)}\\
&= Tx + Ty. && \text{(definition of } T\text{)}
\end{aligned}$$

And for $\alpha \in \mathbb{R}$ and $x \in X$ we have

$$\begin{aligned}
T(\alpha x) &= \lim T_n(\alpha x) && \text{(definition of } T\text{)}\\
&= \lim \alpha T_n x && \text{(each } T_n \text{ is linear)}\\
&= \alpha \lim T_n x && \text{(continuity of scalar multiplication)}\\
&= \alpha Tx, && \text{(definition of } T\text{)}
\end{aligned}$$

thus T is linear.

To see that T is bounded, let $x \in X$, and note that

$$\|Tx\| \leq \sup\{\|T_n x\|\} \leq \sup\{\|T_n\|\|x\|\} = \sup\{\|T_n\|\}\|x\|.$$

Since $\{T_n\}$ is a Cauchy sequence it is necessarily bounded (by the triangle inequality), thus $\|T\|$ is bounded by $\sup\{\|T_n\|\}$ and $T \in \mathcal{B}(X, Y)$.

Lastly, to see that $T_n \to T$, let $\epsilon > 0$ and $N \in \mathbb{N}$ such that $\|T_n - T_m\| < \epsilon$ whenever $m, n > N$. Then for $x \in X$ and $n > N$,

$$\|(T_n - T)x\| = \|T_n x - Tx\| = \lim \|T_n x - T_m x\| \leq \lim \|T_n - T_m\|\|x\| \leq \epsilon\|x\|$$

so $\|T_n - T\| \leq \epsilon$. Since ϵ was arbitrary, we conclude $T_n \to T$. □

Proposition 11. *Let X be a normed vector space. For $x \in X$, the function $\widehat{x} : X^* \to \mathbb{R}$ defined by $\langle \widehat{x}, x^* \rangle = \langle x^*, x \rangle$ is continuous and linear and $\|\widehat{x}\| = \|x\|$.*

Proof. The linearity of $\widehat{x}$ follows from the definition since X^* is a vector space. As for $\widehat{x}$ being bounded, for $x^* \in X^*$ and $x \in X$, the calculation

$$|\langle \widehat{x}, x^* \rangle| = |\langle x^*, x \rangle| \leq \|x^*\|\|x\| = \|x\|\|x^*\| \text{ shows that } \|\widehat{x}\| \leq \|x\|.$$

Clearly $\|\widehat{x}\| = \|x\|$ when $x = 0$, so let $x \in X \setminus \{0\}$. We will find a linear functional $\bar{x}^* \in \mathbb{S}^*$ such that $\langle \widehat{x}, \bar{x}^* \rangle = \|x\|$ which will mean $\|\widehat{x}\| \geq \|x\|$.

First, define a linear functional x^* on $W = \operatorname{span}\{x\}$ by $\langle x^*, \alpha x \rangle = \alpha\|x\|$. Now x^* is bounded, for $|\langle x^*, \alpha x \rangle| = |\alpha|\|x\| = \|\alpha x\|$ so $\|x^*\| = 1$, and note as well that $|\langle x^*, x \rangle| = \|x\|$. The Hahn-Banach theorem provides the existence of a bounded linear functional $\bar{x}^* \in X^*$ such that $\|\bar{x}^*\| = \|x^*\| = 1$ and $\bar{x}^*|_W = x^*$. This gives

$$\langle \widehat{x}, \bar{x}^* \rangle = \langle \bar{x}^*, x \rangle = \|x\|,$$

as desired. □

The next definition provides examples of topologies generated by families of functions per Proposition 6.

Definition 12. Let X be a normed vector space. The WEAK TOPOLOGY $\sigma(X, X^*)$ on X is the weak topology generated X^*. The WEAK-STAR TOPOLOGY $\sigma(X^*, X)$ on X^* is the weak topology generated by the family of functions $\{\widehat{x} : x \in X\}$ from Proposition 11.

Proposition 13. *A sequence $\{x_n\}$ in a normed vector space X converges weakly to $\bar{x} \in X$ if and only if $\langle x^*, x_n \rangle \to \langle x^*, \bar{x} \rangle$ for all $x^* \in X^*$. Similarly, a sequence $\{x_n^*\}$ in the dual of a normed vector space X converges in the weak-star topology to $\bar{x}^* \in X^*$ if and only if $\langle \widehat{x}, x_n^* \rangle \to \langle \widehat{x}, \bar{x}^* \rangle$.*

Proof. This follows from Proposition 7. $\square$

If we identify X with it's natural embedding in X^{**}, Proposition 11 shows that $X = \{\widehat{x} : x \in X\} \subseteq X^{**}$. It is not always the case that these are equivalent; when they are:

Definition 14. A Banach space X is REFLEXIVE if $\{\widehat{x} : x \in X\} = X^{**}$.

As X^* is a Banach space, it too can be reflexive or not as a subset of X^{***} via the isometries $x^* \mapsto \widehat{x^*}$ where for $x^{**} \in X^{**}$, $\langle \widehat{x^*}, x^{**} \rangle = \langle x^{**}, x^* \rangle$.

Conway points to a reference in [2, p.89] showing that there are Banach spaces that are isometric to their second dual that are *not* reflexive. Reflexivity requires them to be isometrically isomorphic via the natural embedding $x \mapsto \widehat{x}$.

Example 15. For $1 < p < \infty$ and a σ-finite measure space X, the Banach spaces $L^p(X)$ are reflexive.

Also, via the Riesz representation theorem, a Hilbert space $\mathcal{H}$ can be identified with its dual $\mathcal{H}^*$ thus Hilbert spaces are reflexive.

Example 16. A often cited example of a non-reflexive Banach space is c_0 (sequences that vanish at infinity), for $c_0^* = \ell^1$ (absolutely summable sequences) and $(\ell^1)^* = \ell^\infty$ (bounded sequences), thus $c_0 \subsetneq (c_0)^{**} = \ell^\infty$ via the natural embedding.

Proposition 17. *For a Banach space X, the following are equivalent:*

(a) *X is reflexive.*

(b) *X^* is reflexive.*

(c) *$\sigma(X^*, X) = \sigma(X^*, X^{**})$ (the weak-star topology on X^* is the same as the weak topology on X^* generated by X^{**}).*

(d) *The closed unit ball in X is weakly compact.*

Proof. See [2, Theorem 4.2]. $\square$

3 Convex Sets and Functions [3]

Definition 18. A non-empty subset Ω of a vector space X is CONVEX if whenever $x, y \in \Omega$ and $\lambda \in (0, 1)$, we have $\lambda x + (1 - \lambda)y \in \Omega$.

Given a vector space X we regard $X \times \mathbb{R}$ as a real vector space with pointwise addition and scalar multiplication. We now provide a geometrical definition of a convex function via its epigraph.

Definition 19. Let X be a non-empty set and $f : X \to \mathbb{R}$. The EPIGRAPH of f is the set

$$\operatorname{epi} f = \{(x, \lambda) \in X \times \mathbb{R} : f(x) \leq \lambda\}.$$

Definition 20. Let X be a real vector space. A function $f : X \to \mathbb{R}$ is CONVEX if $\operatorname{epi} f$ is a convex subset of $X \times \mathbb{R}$.

The geometrical definition of a convex function via the epigraph is equivalent to a characterization of function convexity using Jensen's inequality.

Proposition 21. *A function $f : X \to \mathbb{R}$ on a real vector space X is convex if and only if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \text{ for all } x, y \in X \text{ and } \lambda \in (0,1). \tag{1}$$

Proof. This is a direct application of the definitions. Suppose $f : X \to \mathbb{R}$ is convex so $\operatorname{epi} f$ is convex, let $x, y \in X$, and let $\lambda \in (0,1)$. Then $(x, f(x)), (y, f(y)) \in \operatorname{epi} f$ so

$$\begin{aligned} \lambda(x, f(x)) + (1-\lambda)(y, f(y)) &= (\lambda x, \lambda f(x)) + ((1-\lambda)y, (1-\lambda)f(y)) \\ &= (\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda)f(y)) \end{aligned}$$

also belongs to $\operatorname{epi} f$, or $f(\lambda x + (1-\lambda)y \leq \lambda f(x) + (1-\lambda)f(y)$.

Conversely, suppose (1) holds, let $(x, \alpha), (y, \beta) \in \operatorname{epi} f$, and let $\lambda \in (0,1)$. We first write the resulting convex combination of the two arbitrary points from $\operatorname{epi} f$:

$$\begin{aligned} \lambda(x, \alpha) + (1-\lambda)(y, \beta) &= (\lambda x, \lambda \alpha) + ((1-\lambda)y, (1-\lambda)\beta) \\ &= (\lambda x + (1-\lambda)y, \lambda \alpha + (1-\lambda)\beta). \end{aligned} \tag{2}$$

We now apply (1) and the facts that $f(x) \leq \alpha$ and $f(y) \leq \beta$ obtaining

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \leq \lambda \alpha + (1-\lambda)\beta$$

which shows that the expression in (2) belongs to $\operatorname{epi} f$, or $\operatorname{epi} f$ is convex. □

We will use the notion of sublevel sets later in the presentation when discussing lower semicontinuous functions.

Definition 22. Given a non-empty set X, a function $f : X \to \mathbb{R}$, and a number $\alpha \in \mathbb{R}$, the SUBLEVEL SET $\mathcal{L}_\alpha$ is the set

$$\mathcal{L}_\alpha = \{x \in X : f(x) \leq \alpha\}.$$

Proposition 23. *Given a real vector space X, a convex function $f : X \to \mathbb{R}$, and a number $\alpha \in \mathbb{R}$, the sublevel set $\mathcal{L}_\alpha$ is always convex.*

Proof. Let $\alpha \in \mathbb{R}$, $x, y \in \mathcal{L}_\alpha$, and $\lambda \in (0,1)$. Then

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\leq \lambda f(x) + (1-\lambda)f(y) && (f \text{ is convex}) \\ &\leq \lambda \alpha + (1-\lambda)\alpha && (x, y \in \mathcal{L}_\alpha) \\ &= \alpha, \end{aligned}$$

so $\lambda x + (1-\lambda)y \in \mathcal{L}_\alpha$. Therefore $\mathcal{L}_\alpha$ is convex. □

We will need the notion of convex separation for a subsequent result.

Definition 24. Let X be a real normed vector space and $\Omega_1, \Omega_2 \subseteq X$. The sets Ω_1 and Ω_2 can be SEPARATED BY A CLOSED HYPERPLANE if there is a non-zero continuous linear functional $x^* \in X^*$ such that

$$\sup\{\langle x^*, x\rangle : x \in \Omega_1\} \leq \inf\{\langle x^*, x\rangle : x \in \Omega_2\}.$$

The sets Ω_1 and Ω_2 can be PROPERLY SEPARATED BY A CLOSED HYPERPLANE if they can be separated by a continuous linear functional $x^* \in X^*$ and

$$\inf\{\langle x^*, x\rangle : x \in \Omega_1\} < \sup\{\langle x^*, x\rangle : x \in \Omega_2\}.$$

And lastly, the sets Ω_1 and Ω_2 can be STRICTLY SEPARATED BY A CLOSED HYPERPLANE if there is a continuous linear functional $x^* \in X^*$ such that

$$\sup\{\langle x^*, x\rangle : x \in \Omega_1\} < \inf\{\langle x^*, x\rangle : x \in \Omega_2\}.$$

The notion of convex separation is closely related to the Hahn-Banach theorem, which is typically invoked for proving convex separation results. Also, it is possible to use convex separation to prove the Hahn-Banach theorem [3, Theorem 2.54]. We will use the following convex separation result to show that norm-closed convex sets are weakly closed.

Proposition 25. *Let X be a normed vector space and let $\Omega_1, \Omega_2 \subseteq X$ be non-empty, disjoint, convex sets such that Ω_1 is closed and Ω_2 is compact. Then Ω_1 and Ω_2 can be strictly separated by a closed hyperplane.*

Proof. See [3, Theorem 2.61]. □

4 The Weak Topology and its Interplay with Convexity [3]

We collect a few interesting results here demonstrating some differences between the weak and norm topologies in infinite dimensional normed vector spaces as well as similarities in the presence of convexity. We will use the notation $\mathbb{B}$ for the closed unit ball, B for the open unit ball, $\mathbb{B}^*$ for the closed unit ball in the dual space, $\mathbb{S}$ for the unit sphere, and $\overline{\mathbb{S}}^w$ for the weak-closure of $\mathbb{S}$.

We first apply Proposition 7 to show how weak and norm topologies can differ in infinite-dimensional settings.

Proposition 26. *Any infinite orthonormal sequence in an infinite-dimensional Hilbert space converges weakly to 0.*

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in an infinite-dimensional Hilbert space $\mathcal{H}$. We will show that $\langle h^*, e_n\rangle \to 0$ for any bounded linear functional h^* on $\mathcal{H}$. To that end,

$$\begin{aligned} \|h^*\|^2 &\geq \sum_n \|\langle h^*, e_n\rangle e_n\|^2 && \text{(Bessel's inequality)} \\ &= \sum_n |\langle h^*, e_n\rangle|^2 \end{aligned}$$

which means $\langle h^*, e_n\rangle \to 0$. Since h^* was arbitrary, we conclude $e_n \to 0$ weakly. □

Proposition 27. *In an infinite-dimensional normed vector space we have* $\overline{\mathbb{S}}^w = \mathbb{B}$, *thus the unit sphere is never weakly closed.*

Proof. Let X be in infinite-dimensional normed vector space. Note first that $x \in \mathbb{B}$ if and only if $|\langle x^*, x\rangle| \leq 1$ whenever $x^* \in \mathbb{B}^*$, or

$$\mathbb{B} = \bigcap_{x^* \in \mathbb{B}^*} \{x \in X : |\langle x^*, x\rangle| \leq 1\}.$$

Namely, $\mathbb{B}$ is an intersection of weakly closed sets thus is itself weakly closed. Since $\mathbb{S} \subseteq \mathbb{B}$, we have $\overline{\mathbb{S}}^w \subseteq \mathbb{B}$.

For the reverse inclusion, we will show $B \subseteq \overline{\mathbb{S}}^w$. Since $\mathbb{B} = B \cup \mathbb{S}$ and $\mathbb{S} \subseteq \overline{\mathbb{S}}^w$, we will have $\mathbb{B} \subseteq \overline{\mathbb{S}}^w$ as desired. To that end, let $\bar{x} \in B$ and let U be a weakly open set containing $\bar{x}$. In that case, there is a basic open set G of the form

$$G = \{x \in X : |\langle x_i^*, x - \bar{x}\rangle| < \epsilon,\ i = 1, \dots, n\}$$

for some $x_1^*, \dots, x_n^* \in X^*$ and $\epsilon > 0$ which satisfies $\bar{x} \in G \subseteq U$. Note that there must be a non-zero $\bar{y} \in X$ such that $\langle x_i^*, \bar{y}\rangle = 0$ for $i = 1, \dots, n$, otherwise the linear mapping $X \to \mathbb{R}^n$ defined by $x \mapsto (\langle x_1^*, x\rangle, \dots, \langle x_n^*, x\rangle)$ would be injective which it can not be since $\dim X = \infty$.

Now, note that

$$\langle x_i^*, (\bar{x} + t\bar{y}) - \bar{x})\rangle = t\langle x_i^*, \bar{y}\rangle = 0$$

for all $t \in \mathbb{R}$, thus $\bar{x} + t\bar{y} \in G$ for all $t \in \mathbb{R}$. Also, for $t \in \mathbb{R}$ the range of the continuous function $t \mapsto \|\bar{x} + t\bar{y}\|$ contains $[\|\bar{x}\|, \infty)$, thus there is a number $\bar{t}$ such that $\|\bar{x} + \bar{t}\bar{y}\| = 1$, or

$$\bar{x} + \bar{t}\bar{y} \in \mathbb{S} \cap G \subseteq \mathbb{S} \cap U.$$

We have shown that every weakly open set U containing $\bar{x} \in B$ has a nonempty intersection with $\mathbb{S}$, or $B \subseteq \overline{\mathbb{S}}^w$ as desired. □

Similarly, the open unit ball is never weakly open in an infinite-dimensional normed space.

Proposition 28. *In an infinite-dimensional normed space, the open unit ball* B *is never weakly open.*

Proof. Suppose B is weakly open in an infinite-dimensional normed space X. In that case B^c is weakly closed. Also $\mathbb{B}$ is weakly closed, which would mean $\mathbb{S} = \mathbb{B} \cap B^c$, being the intersection of of two weakly closed sets, is also weakly closed which it is not per the previous proposition. We thus conclude that B can not be weakly open. □

Proposition 27 shows that the closed unit ball is weakly closed. More can be said, and the following result will be important for this presentation.

Proposition 29. *A closed convex subset of a normed vector space is weakly closed.*

Proof. Let X be a normed vector space, let $\Omega \subseteq X$ be closed and convex, and let $\bar{x} \in \Omega^c$. By Proposition 25, Ω and $\{\bar{x}\}$ can be strictly separated by a closed hyperplane. That is, there is an $\bar{x}^* \in X^*$ and a number $\lambda \in \mathbb{R}$ such that

$$\langle \bar{x}^*, x \rangle < \lambda < \langle \bar{x}^*, \bar{x} \rangle \text{ for all } x \in \Omega.$$

Then $U := \{x \in X : \langle \bar{x}^*, x \rangle > \lambda\}$ is a weakly open set satisfying $\bar{x} \in U \subseteq \Omega^c$, therefore Ω^c is weakly open so Ω is weakly closed. □

Theorem 30 (Alaoglu's theorem)**.** *The closed unit ball in the dual of a normed vector space is compact in the weak-star topology.*

Proof. See [3, Corollary 1.113]. □

Note that for a reflexive Banach space X, Theorem 30 means the closed unit ball in X^* is weakly compact, and X^* is also reflexive so the closed until ball in $X^{**} = X$ is weakly compact (as indicated in Proposition 17).

5 Lower Semicontinuous Functions and Minimizers [3]

We now arrive at the final section where we will apply the above material to arrive at assertions concerning the existence of minimizers for lower semicontinuous functions.

Definition 31. Let X be a topological space. A function $f : X \to \mathbb{R}$ is LOWER SEMICONTINUOUS (or lsc) at $\bar{x} \in X$ if for every $\alpha < f(\bar{x})$ there is an open set U containing $\bar{x}$ such that $\alpha < f(x)$ for all $x \in U$.

Lower semicontinuity of a function can be equivalently characterized via its epigraph and also its sublevel sets.

Proposition 32. *Let X be a topological space. For a function $f : X \to \mathbb{R}$, the following are equivalent*

(a) f is lower semicontinuous.

(b) $\operatorname{epi} f$ is closed (as a subset of the product topological space $X \times \mathbb{R}$).

(c) The sublevel set $\mathcal{L}_\alpha$ is closed for all $\alpha \in \mathbb{R}$.

Proof. (a) $\implies$ (b) Suppose $f : X \to \mathbb{R}$ is lsc and let $(\bar{x}, \alpha) \in (\operatorname{epi} f)^c$. This gives $\alpha < f(\bar{x})$, in which case there is a number $\epsilon > 0$ such that $\alpha + \epsilon < f(\bar{x})$. Since f is lsc, there is an open set U such that $\bar{x} \in U$ and $f(x) > \alpha + \epsilon$ for all $x \in U$. This means

$$(\bar{x}, \alpha) \in U \times (\alpha - \epsilon, \alpha + \epsilon) \subseteq (\operatorname{epi} f)^c$$

so $(\operatorname{epi} f)^c$ is open, thus $\operatorname{epi} f$ is closed.

(b) $\implies$ (c) Suppose epi f is closed, let $\alpha \in \mathbb{R}$, and let $\bar{x} \in (\mathcal{L}_\alpha)^c$. Since $\bar{x} \notin \mathcal{L}_\alpha$, we have $\alpha < f(\bar{x})$ or $(\bar{x}, \alpha) \in (\text{epi}\, f)^c$. Since $(\text{epi}\, f)^c$ is open, there is an open set $U \subseteq X$ containing $\bar{x}$ and an $\epsilon > 0$ such that

$$U \times (\alpha - \epsilon, \alpha + \epsilon) \subseteq (\text{epi}\, f)^c, \text{ or } f(x) > \alpha + \epsilon > \alpha \text{ for all } x \in U.$$

Thus $\bar{x} \in U \subseteq (\mathcal{L}_\alpha)^c$ so $(\mathcal{L}_\alpha)^c$ is open, or $\mathcal{L}_\alpha$ is closed.

(c) $\implies$ (a) Suppose $\mathcal{L}_\alpha$ is closed for all $\alpha \in \mathbb{R}$, let $\bar{x} \in X$, and let $\alpha < f(\bar{x})$. In that case $(\bar{x}, \alpha) \in (\mathcal{L}_\alpha)^c$ which is open, so there is an open set U such that $\bar{x} \in U \subseteq (\mathcal{L}_\alpha)^c$. This means $f(x) > \alpha$ for all $x \in U$, or f is lsc. □

Lower semicontinuity and weak lower semicontinuity are equivalent for convex functions. This holds in the more general topological space setting but we will prove it here for normed vector spaces applying previous results from this presentation.

Proposition 33. *Let X be a real normed vector space. A convex function $f : X \to \mathbb{R}$ is lower semicontinuous if and only if it is weakly lower semicontinuous.*

Proof. Let $f : X \to \mathbb{R}$ be a convex function on a normed vector space X. If f is lsc, then epi f is closed by Proposition 32. Thus epi f is closed and convex, therefore it is weakly closed by Proposition 29. Another application of Proposition 32 shows that f is weakly lsc.

Conversely, if f is convex and weakly lsc, then epi F convex and weakly closed thus is closed, so f is lsc by Proposition 32. □

We now arrive at our first result for minimizers in the context of lower semicontinuous functions on topological spaces.

Proposition 34. *Let X be a topological space and $f : X \to \mathbb{R}$ a lower semicontinuous function. If there is a number $\alpha > \inf\{f(x) : x \in X\}$ such that $\mathcal{L}_\alpha$ is compact, then f has an absolute minimizer in X. That is, there is a point $\bar{x} \in X$ such that $f(\bar{x}) = \inf\{f(x) : x \in X\}$.*

Proof. Let $f : X \to \mathbb{R}$ be lsc, and suppose $\alpha > \inf\{f(x) : x \in X\}$ with $\mathcal{L}_\alpha$ compact. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of numbers such that $\inf\{f(x) : x \in X\} \leq \alpha_n \leq \alpha$ for all n and $\alpha_n \to \inf\{f(x) : x \in X\}$. Since $\alpha_{n+1} \leq \alpha_n \leq \alpha$ for all n, we have $\mathcal{L}_{\alpha_{n+1}} \subseteq \mathcal{L}_{\alpha_n} \subseteq \mathcal{L}_\alpha$ for all n. Since f is lsc, each $\mathcal{L}_{\alpha_n}$ is a closed subset of the compact set $\mathcal{L}_\alpha$, thus each $\mathcal{L}_{\alpha_n}$ is compact. By [3, Lemma 2.162], $\cap_n \mathcal{L}_{\alpha_n}$ is non-empty, so choose $\bar{x} \in \cap_n \mathcal{L}_{\alpha_n}$. Then

$$\inf\{f(x) : x \in X\} \leq f(\bar{x}) \leq \alpha_n \text{ for all } n \text{ and } \alpha_n \to \inf\{f(x) : x \in X\},$$

so $f(\bar{x}) = \inf\{f(x) : x \in X\}$ as desired. □

And for the final result, the above applies in the context of reflexive Banach spaces if a similar sublevel set is merely bounded rather tha compact. This result is an analogue to the extreme value theorem, which provides that a continuous function on a closed and bounded interval (or more generally, on a compact subset of a topological space) attains a max and a min.

Theorem 35. *Let X be a reflexive Banach space and $f : X \to \mathbb{R}$ a convex lower semicontinuous function. If there is a number $\alpha > \inf\{f(x) : x \in X\}$ such that $\mathcal{L}_\alpha$ is bounded, then f has an absolute minimizer in X. That is, there is a point $\bar{x} \in X$ such that $f(\bar{x}) = \inf\{f(x) : x \in X\}$.*

Proof. Let X be a reflexive Banach space, $f : X \to \mathbb{R}$ convex and lsc, and $\alpha > \inf\{f(x) : x \in X\}$ such that $\mathcal{L}_\alpha$ is bounded. Since f is lsc and convex, f is weakly lsc by Proposition 33, and $\mathcal{L}_\alpha$ is weakly closed by Proposition 32. Since X is reflexive and $\mathcal{L}_\alpha$ is weakly closed and bounded, it is compact, and the conclusion follows from Proposition 34. □

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