On the discontinuities of derivatives.

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April 12, 2019

Abstract

The absolute value function is often cited when discussing functions that are not everywhere differentiable. Less often mentioned are functions that are everywhere differentiable, but for which the resulting derivative function fails to be continuous. The canonical example of such a function is

$$f(x) = x^2 \sin(1/x)$$

where we set $f(0) = 0$. The derivative at zero is defined and is equal to zero, but the slope of the tangent lines oscillate between -1 and 1 ever faster as we approach zero.

In this talk, we'll discuss just how badly discontinuous such a derivative can become. We'll begin with a few theorems that show derivatives cannot have so-called jump or removable discontinuities, then go on to explore functions with derivatives that are discontinuous on dense subsets of the real line.

1 Outline

In this talk we will consider functions $f : \mathbb{R} \to \mathbb{R}$ of a single real variable.

1. Not all such functions have a derivative defined everywhere. This implies that the domain of f' may not be all of $\mathbb{R}$.

   (a) The canonical example: $f(x) = |x|$ is not differentiable at 0.

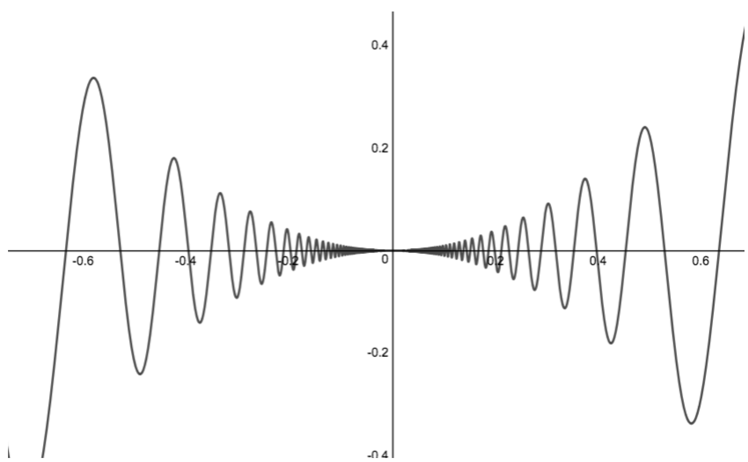
   (b) Much worse examples exist. Abbott gives the example

   $$g(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$$

   where $h(x)$ is the periodic extension of $|x|$ on the interval $[-1, 1]$ [1, p 145]. This function is continuous and nowhere differentiable.

2. Suppose that a function has a derivative defined at all points of its domain. The resulting derivative function need not be continuous.

   (a) Example : $f(x) = x^2 \sin(1/x), \quad f(0) = 0$

   The slope of the line oscillates between -1 and 1 as we approach 0 from either side. But the derivative at zero exists and equals 0.

3. In what ways can derivatives fail to be continuous?

   (a) The "nicest" forms of discontinuity are unavailable to derivatives.

i. From Spivak [3, pg 203] : Suppose that f is continuous at a, and that f' exists in some interval containing a, except perhaps for $x = a$. Suppose, moreover, that $\lim_{x\to a} f'(x)$ exists. Then $f'(a)$ also exists and $\lim_{x\to a} f'(x) = f'(a)$. This says a derivative cannot have a removable discontinuity.

A. The proof uses the mean value theorem. Since f' exists in an interval around a and f is continuous at a, f is continuous on some interval $[a, a+h]$ and differentiable on $(a, a+h)$ for some $h > 0$ (and we can find an $h < 0$ for which a similar, left-sided argument holds). Applying the mean value theorem to f on $[a, a+h]$ we get $\alpha_h \in (a, a+h)$ such that

$$\frac{f(a+h) - f(a)}{h} = f'(\alpha_h)$$

By assumption $\lim_{x\to a^+} f'(x)$ exists, and since $a < \alpha_h < a+h$ for all h, by taking limits above we get that

$$\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0^+} f'(\alpha_h) = \lim_{x\to a^+} f'(x)$$

Since a similar argument holds for $h < 0$, and the limits from above and below agree by assumption, we see that we can define $f'(a)$ and $f'(a) = \lim_{x\to a} f'(x)$

ii. Corollary: If f' is discontinuous at a, then at least one of the one sided limits $\lim_{x\to a^\pm} f'(x)$ must fail to exist[1]. This implies a derivative cannot exhibit a jump discontinuity.

(b) Thus, the function in the example above displays the simplest form of discontinuity available to a derivative.

4. Although discontinuous derivatives are badly behaved (in the above sense), they satisfy a property seemingly exclusive to continuous functions.

(a) Darboux's Theorem states that if f is differentiable on $[a, b]$ and there exists c such that $f'(a) < c < f'(b)$, then there exists $x \in (a, b)$ with $f'(x) = c$.

(b) The latter is called the "intermediate value property." The intermediate value theorem states that all continuous functions have this property. Darboux's Theorem states that all derivatives, no matter how discontinuous, still have the Intermediate Value Property.

(c) Can a function with a jump or removable discontinuity have the intermediate value property?

[1]If both exist and are equal, f' is continuous at a, while if they both exist and are unequal, $f'(a)$ cannot be defined (right and left difference quotients associate to upper and lower limits via MVT), so f' is not defined at a, hence cannot be discontinuous there.

5. How discontinuous can a derivative be?

   (a) Any derivative must have points of continuity. Thus the Dirichlet function cannot be a derivative.

   (b) The above follows due to the fact that the derivative of a function is a pointwise limit of a sequence of continuous functions, namely the sequence defined by

   $$f_n(x) = \frac{f(x + 1/n) - f(x)}{1/n}.$$

   Such functions are known as Baire Category 1 functions. From [4]: The Baire Characterisation Theorem states that a real valued function f defined on a Banach space X is a Baire-1 function if and only if for every non-empty closed subset K of X, the restriction of f to K has a point of continuity relative to the topology of K.

   (c) In fact, this allows us to conclude that a derivative must be continuous on a dense set.

6. We will next construct an example that shows that a derivative may be discontinuous on dense subset of positive measure.

   (a) It is worth convincing ourselves that a function that is zero on one dense set A but positive on another dense set B must be discontinuous at all points in B. If $x \in B$, then $f(x) > 0$: pick $0 < \epsilon < f(x)$. Any neighborhood of x contains elements of A, so for any $\delta > 0$ there is some $y \in A$ with $|x - y| < \delta$ but $|f(x) - f(y)| = |f(x)| > \epsilon$.

   (b) The following is due to Dimitrie Pompeiu, see [2, pg 27]. Let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of the rationals in $[0, 1]$, and $\{A_n\}_{n=1}^{\infty}$ be a sequence of positive reals with $\sum A_n < \infty$. The sums

   $$\sum_{n=1}^{N} A_n \sqrt[3]{(x - q_n)}$$

   converge by the Weirstrass M-Test. It can be shown that the resulting function f is strictly increasing, with finite positive derivative given by the differentiated series (where the differentiated series converges) and infinite otherwise. It can also be shown that f^{-1} is strictly increasing and differentiable, with derivative equal to zero on the set of images of the rational points.

   (c) It remains to show that there is a dense subset of Q^c for which the above derivative is non-zero. This is asserted to be true by Bruckner [2]. Also, see graphics.

   (d) In general, we can create a derivative function that is discontinuous on a preassigned countable set by taking a uniformly convergent series of derivative functions, where each derivative in the associated sequence is discontinuous at a given point in the countable set.

(e) We have seen that derivatives must have at least a dense set of continuity, but that they may exhibit discontinuity on a dense set as well. In fact we have the following Theorem [2, pg 27]: A set $E \subset [a, b]$ is the set of discontinuities of a derivative if and only if it is a countable union of closed sets and is of the first category.

7. The construction due to Abbott [1, pg 207] uses the canonical example from the introduction. Essentially, it places the behavior that example illustrates at 0 at each point of the a Cantor-like set of positive measure.

8. This last example explains why the Fundamental Theorem of Calculus is worded as it is.

   (a) The Fundamental Theorem of Calculus: Suppose f is integrable on $[a, b]$ and $f = g'$ for some function g. Then
   $$\int_a^b f = g(b) - g(a)$$

   (b) Not the Fundamental Theorem of Calculus: Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then
   $$\int_a^b f' = f(b) - f(a)$$

   (c) Theorem: A function f is Riemann integrable if and only if the set of discontinuities of f has measure zero.

   (d) Since the function f constructed by Abbot has a derivative with a discontinuity set of positive measure, $\int f'dx$ is not defined, so the Not Fundamental Theorem of Calculus does not hold for f when using the Riemann integral.

9. The generalized Riemann integral corrects the issue.

References

[1] Stephen Abbot. *Understanding Analysis*. Springer, 2001.

[2] A. M. Bruckner and J. L. Leonard. Derivatives. *The American Mathematical Monthly*, 73(4):24–56, 1966.

[3] Michael Spivak. *Calculus, 4th ed.* Publish of Perish Inc., 2008.

[4] Wikipedia contributors. Baire function — Wikipedia, the free encyclopedia, 2018. [Online; accessed 12-April-2019].