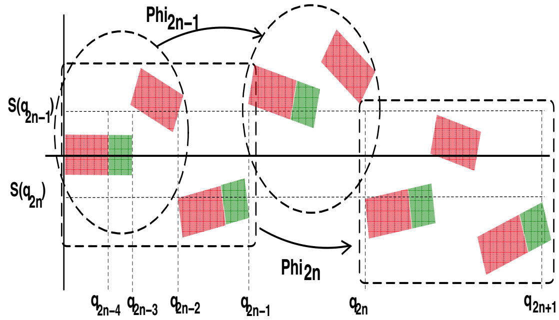
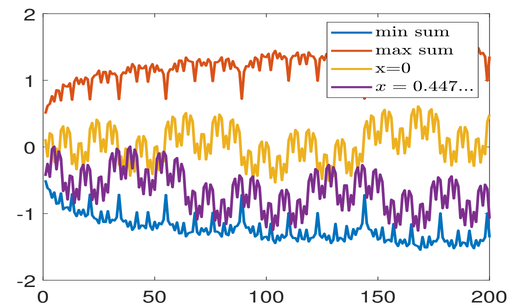
University of Toronto, Canada, Apr 2024





Birkhoff Sums
A Survey of Some Recent Research

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Based on informal Portland State Notes:
Co-authored with L. Fox, F. M. Tangerman, H. Kravitz, C. Aagaard, P. Miracle, I. Shankar, D. Ralston, H. Moore, and others.

Presenting ***Numbers from All Angles***.
Graduate number theory text (under review)
Send inquiries to above email.

SUMMARY:

* We explain the connection between Birkhoff Sums and various subbranches of mathematics: numerical analysis, number theory, ergodic theory, and dynamical systems.

* We prove that one can use Birkhoff sums to measure exactly the *discrepancy* of irrational rotations. Discrepancy was defined by Pisot and Van Der Corput in the 1930s, which is used to study the well-distributedness of infnite sequences in $[0, 1)$.

* We exhibit the surprising variety of measures associated with Birkhoff sums and show that they tile $\mathbb{R}$.

* We conjecture exact growth rates of certain Birkhoff sums for rotations by 'metallic means' (or $\rho = [a, a, \cdots]$). We prove these results for $a = 1$ and $a = 2$.

* We will suppress technical details to keep the exposition accessible for graduate students.

NOTATION:

$$\{x\} \text{ means fractional part of } x$$
$$\lfloor x \rfloor \text{ means integer part of } x$$

The Set Up

Measuring the Sums

Birkhoff Measures

Discrepancy

Computing Birkhoff Measures

Examples of Exact Birkhoff Sums

Partial Proof

Appendix: More Pictures

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INTRODUCING THE BIRKHOFF SUMS

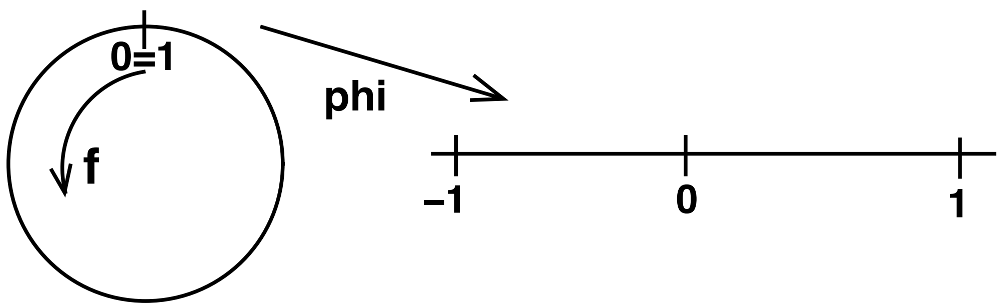
Overisel (Michigan) was named after the Dutch province of Overijssel and is the birthplace of George David Birkhoff.

Definitions

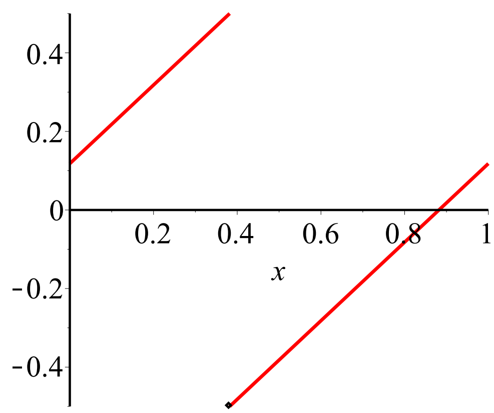
Definition Birkhoff sum. Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ a continuous map and $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ a map with average zero. Then $\boldsymbol{S(f,n,x) := \sum_{i=1}^{n} \phi(f^i(x))}$.

In practice: rotation and "identity" minus 1/2:

$$f_\rho(x) := x + \rho \mod 1 \quad \text{and} \quad \phi(x) = x - 1/2\,.$$



So $\phi(f)$ looks like this:



In this case, Birkhoff sum becomes

$$\boldsymbol{S(\rho,n,x) := \sum_{i=1}^{n} (f_\rho^i(x) - 1/2)}\,.$$

We are interested in the properties of this sum for irrational ρ.

Connections

Birkhoff sums are studied for their own sake, and for their connections with other areas of mathematics.

Connection I: Ergodic Theory is concerned with showing that the **average of S** converges. There is a large literature on how good that convergence is. (Ulcigrai, Bromberg, Dolgopyat, Beck, Sarig, Ralston, Knill, ...).

Connection II: Discrepancy Theory started by Van Der Corput and Pisot in the 1930's. Given an infinite sequence $x := \{x_i\}_{i=1}^{\infty}$ in $[0, 1)$, how **evenly distributed** are the first n points distributed (for any n)? This is important in **Numerical Analysis** to interpolate or estimate a quantity on a set of points generated by a **"low-discrepancy" or "quasirandom" sequence**. This minimizes numerical uncertainties. (wikipedia "Low-discrepancy sequence").

Connection III: Rotations are an important topic in **Dynamical Systems**.

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DIFFERENT WAYS TO MEASURE *S*

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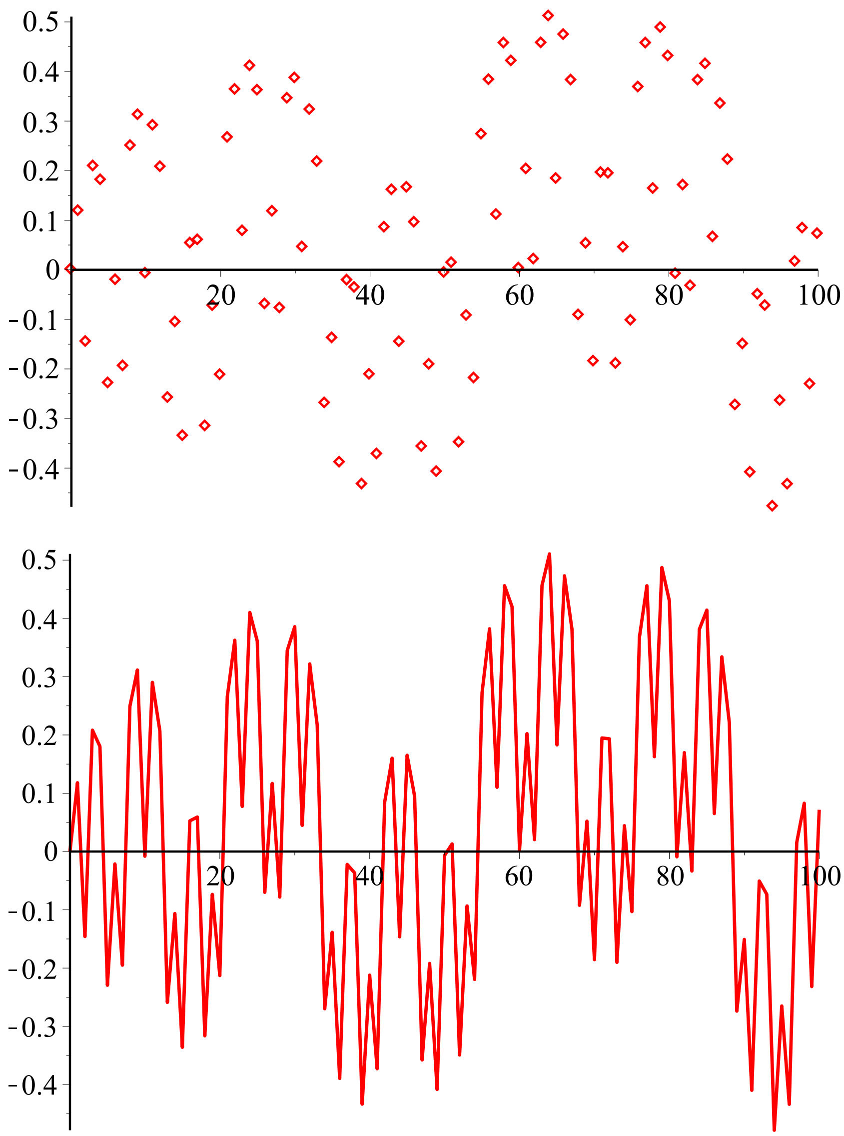
$S(\rho, n, 0)$

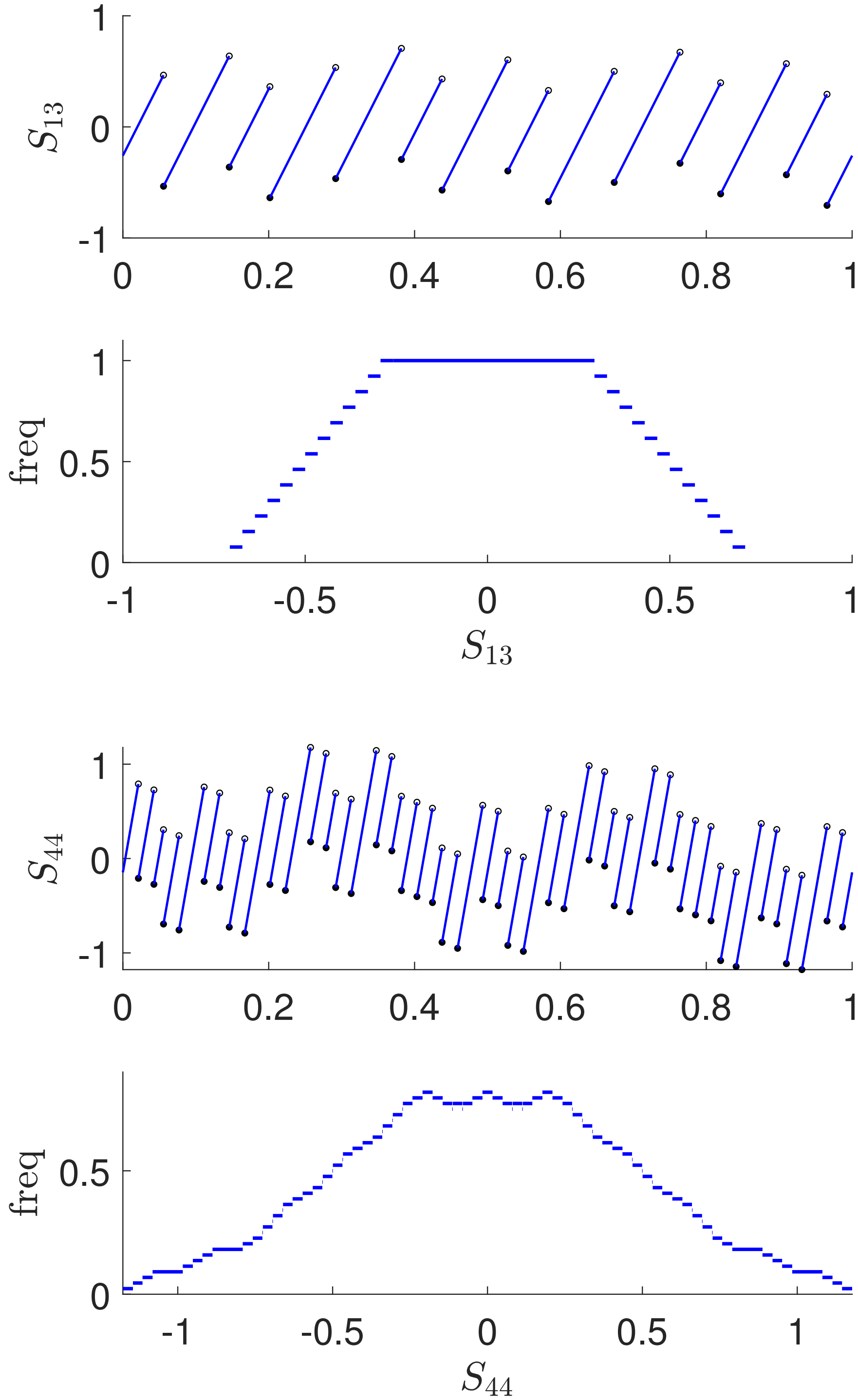
The study of $S(\rho, n, x_0)$ where only n varies. Often we take $x_0 = 0$. Below we plot $S(\rho, n, 0)$ where ρ is the golden mean. We often 'connect the dots' in order to guide the eye.



The "self-similar" structure is evident. Also: wild fluctuations, but envelope grows slowly. In Fact: as $\ln n$.

As a Density

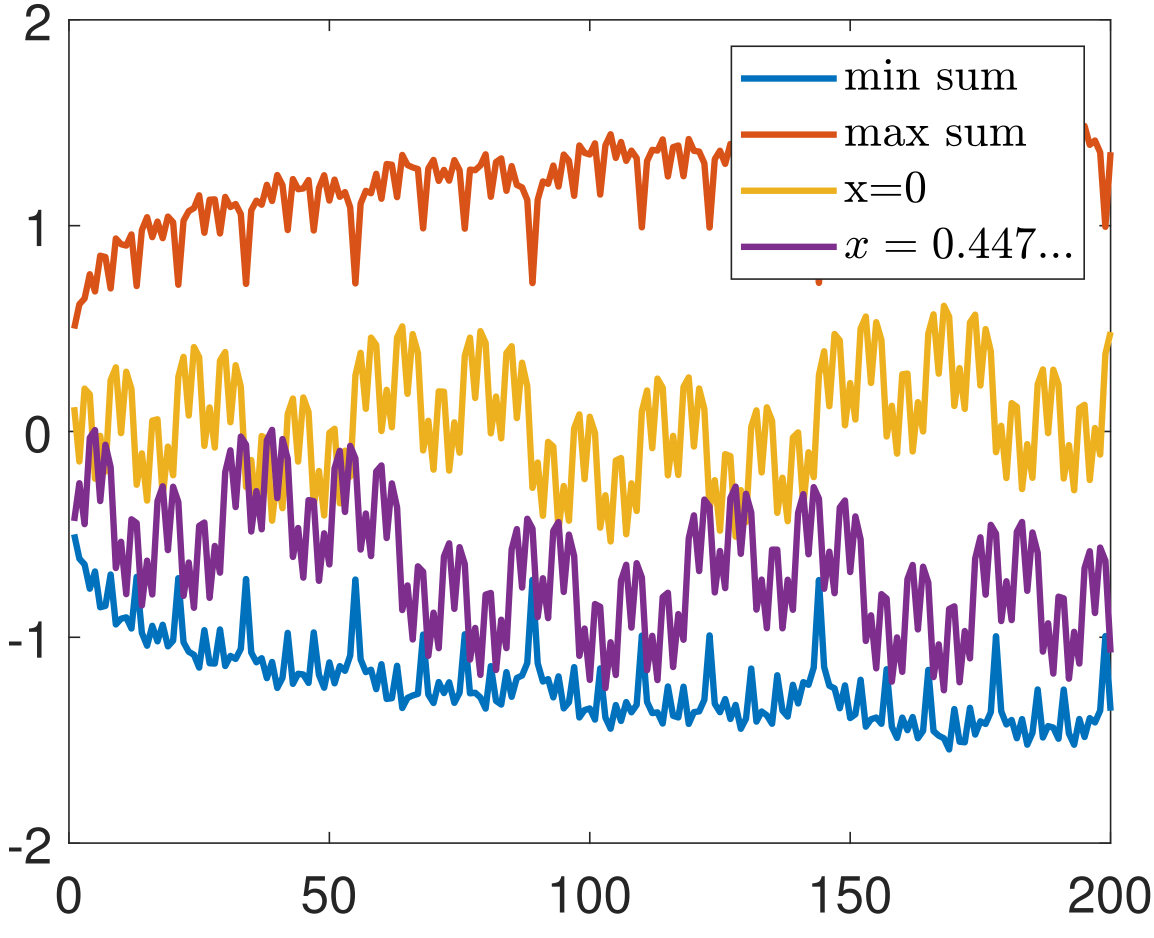
Fix n, study of the measure $\nu(\rho, n, y)$ obtained by taking an infinitesimal interval $I = [y, y + dy)$ in $\mathbb{R}$ and determining its pre-images under $S(\rho, n, x)$. Below, ρ is the golden mean.



The Support of the Density

The support of $\nu(\rho, n, y)$ is an interval $[-M, M]$ and M depends on ρ and n.

Below, ρ is the golden mean. we sketch M (red), $-M$ (blue), and $S(\rho, n, 0)$ (yellow), and $S(\rho, n, 1 - 5^{-1/2})$ (purple).



Clearly $-M \leq S(\rho, n, x_0) \leq M$. But remarkably, the purple signal appears to be **always negative**. This is indeed the the case for an uncountable set of initial conditions x_0 [9]

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A CLOSER LOOK AT THE BIRKHOFF MEASURES

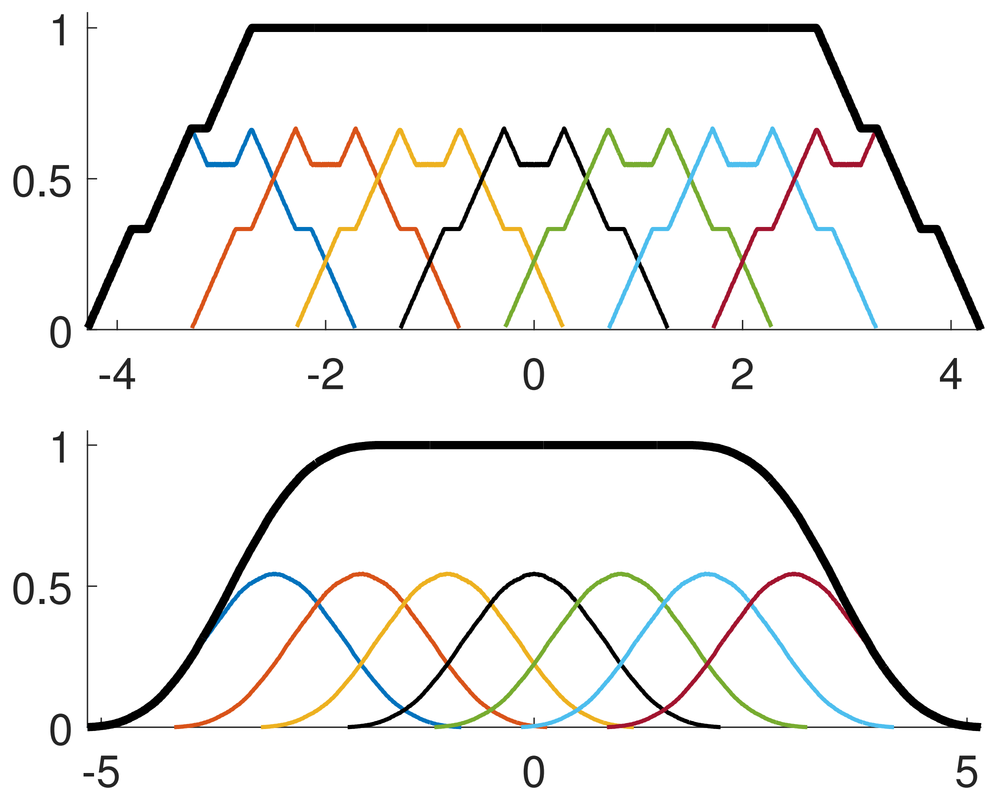
George David Birkhoff

The Densities Tile

These densities are very interesting.

Theorem, symmetries. The densities $\nu(\rho, n, y)$ satisfy:
(i) They are **invariant** under $y \leftrightarrow -y$.
(ii) They are **invariant** under $\rho \leftrightarrow -\rho$.

Theorem, tiling. The densities $\nu(\rho, n, y)$ satisfy:
(i) Their support is a **connected interval**.
(ii) The density is **positive** in the interior of the interval.
(iii) The sum of $\nu(\rho, n, y - i)$ for $i \in \mathbb{Z}$ equals the **Lebesgue measure** on $\mathbb{R}$.



$\rho = e - 2$ and $n = 213$ and 2024, respectively.

Further Properties of the Density

Theorem. If n is a continued fraction denominator of ρ, then ν is an isoceles trapezoid (see page 9).

Theorem [5]. If we average $\nu(\rho, n, z)$ over ρ (with respect the Lebesgue) on $\rho \in [0, 1)$, then the resulting distribution converges to a Cauchy distribution.

However, **for fixed** ρ there is NO convergence as $n \to \infty$. We know that the trapezoid occurs infinitely often. On the next page, we give some idea of the stunning variety of these measures just for $\rho = e - 2$.

For comparison:

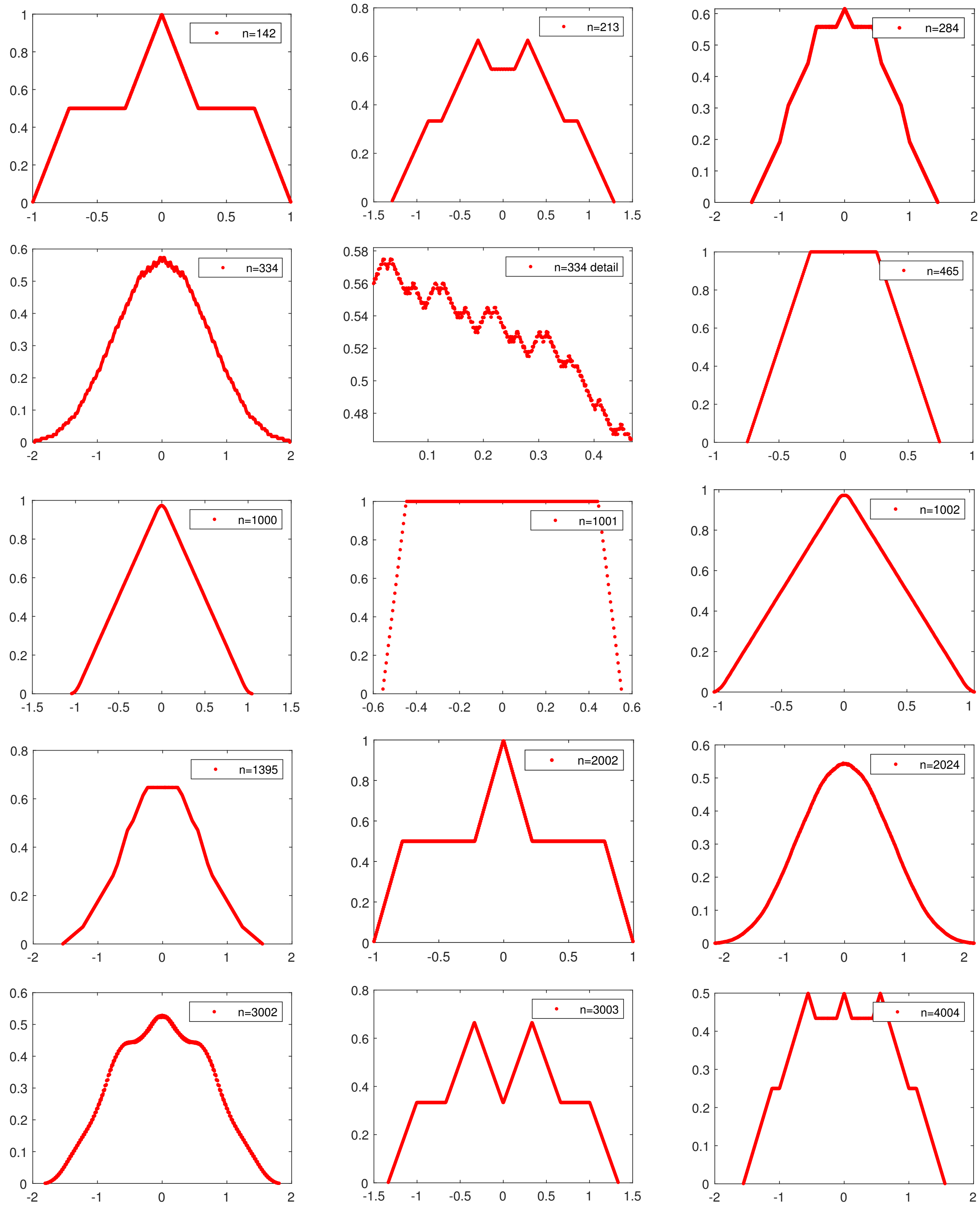
$$e - 2 = [1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \cdots]$$

And the approximants are:

$$1, \frac{2}{3}, \frac{3}{4}, \frac{5}{7}, \frac{23}{32}, \frac{28}{39}, \frac{51}{71}, \frac{334}{465}, \frac{385}{536}, \frac{719}{1001}, \frac{6137}{8544}, \ldots$$

Considering this, we'll consider a *simpler* (?) derived quantity in the next section. The support of the densities is a symmetric closed interval. We'll study the length of that interval.

Illustration of Non-Convergence



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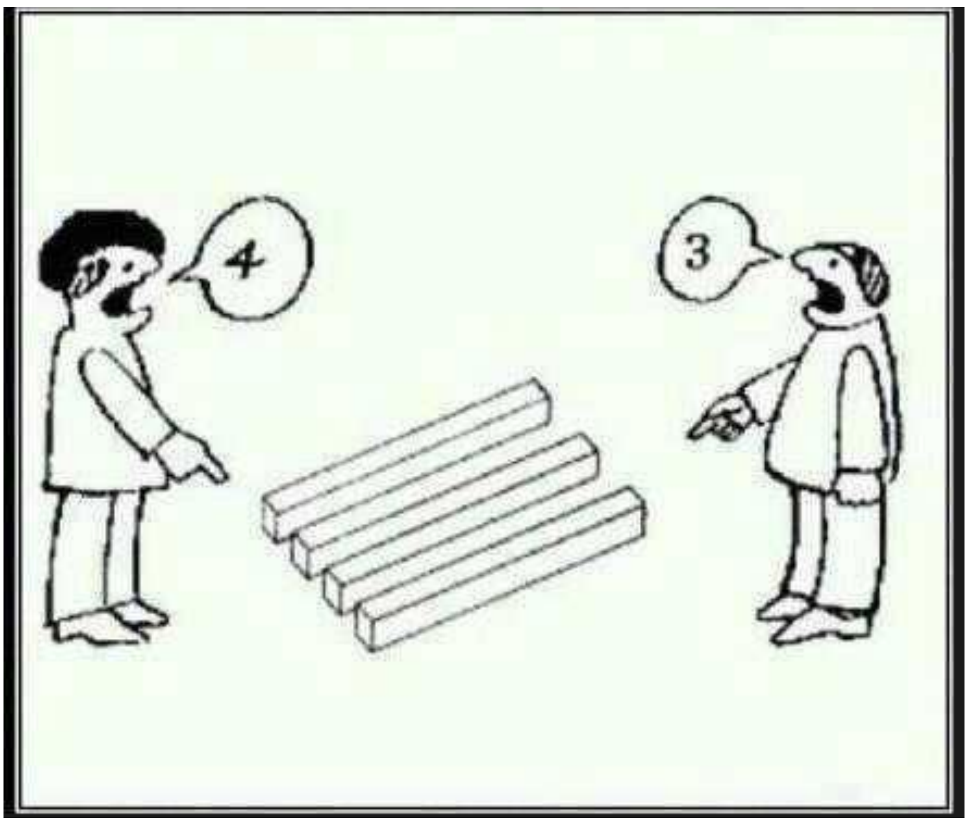
D I S C R E P A N C Y

A N D   T H E   S U P P O R T



Discrepancy

Studied by Pisot and Van Der Corput [4] and later in [7].

Definition. Let $\overline{x} := \{x_i\}_{i=1}^{\infty}$, I an interval in $[0,1)$. Then

$$A(I,n) := \{\#\text{ first } n \text{ points of } \overline{x} \text{ in } I\}.$$

The **discrepancy** $D_n(\overline{x})$ of $\overline{x}$ is

$$D_n(\overline{x}) := \sup_{I \subseteq [0,1)} \left| \frac{A(I,N)}{N} - \ell(I) \right|.$$

Here $I \subseteq [0,1)$ ranges over the half open intervals.
Note: we will use nD_n and call it **clumpiness** $\boldsymbol{C_n(\overline{x})}$.

Example 1. $(x_1, \cdots, x_n) = (x_1, x_1, \cdots, x_1)$: $C_n(\overline{x}) = n$.
Example 2. $(x_1, \cdots, x_n) = (\frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n})$: $C_n(\overline{x}) = 1$.

Intuition: Clumpiness is big if there are underpopulated OR overpopulated intervals. Note that $C_n(\overline{x})$ and $C_{n+1}(\overline{x})$ *cannot* both be perfectly evenly distributed.

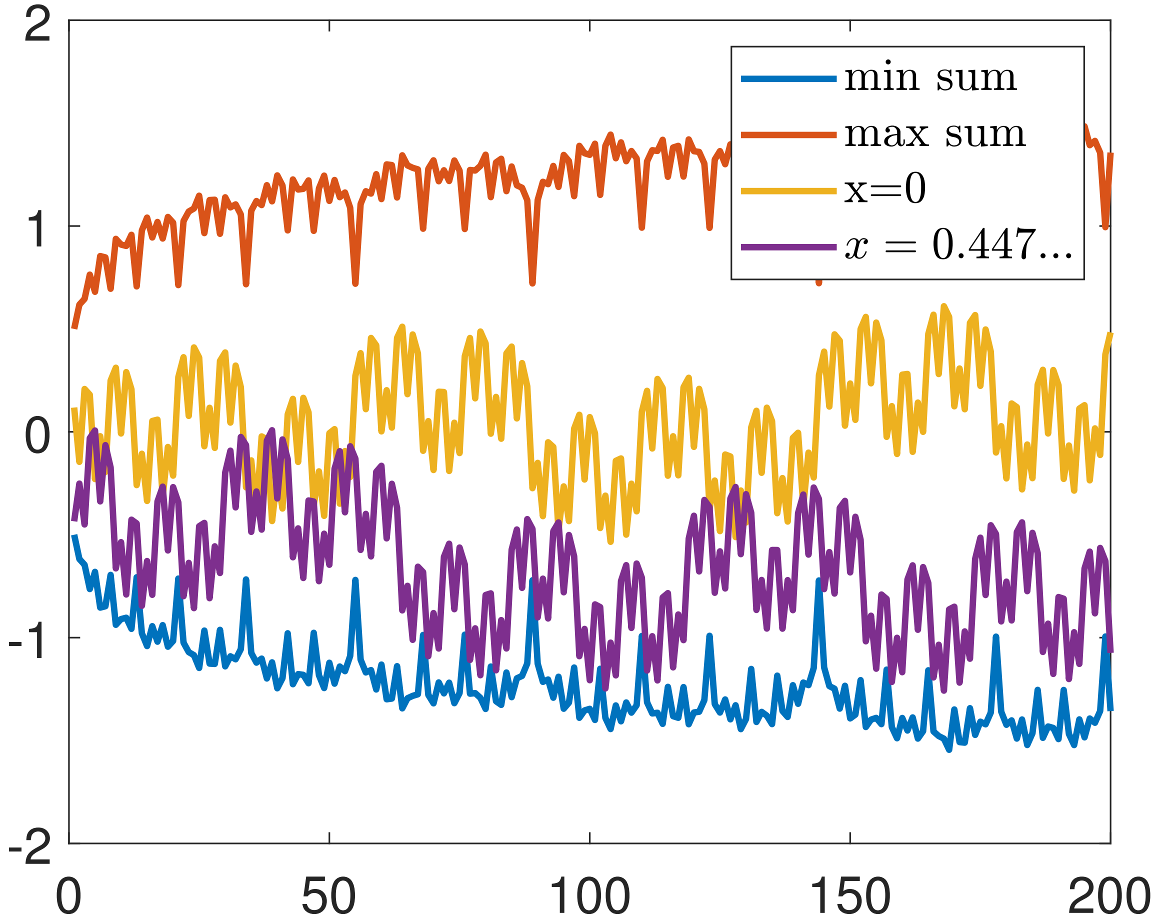
Purpose: Study how evenly distributed an infinite sequence $\overline{x}$ is. Weyl (1916) proved that $\overline{x}$ is uniformly distributed[1] is equivalent to $\lim_{n\to\infty} D_n(\overline{x}) = 0$. Used to generate 'quasi-random' sequences important in numerical analysis.

Theorem. [8] (pg 24) For any infinite sequence $\overline{x}$, the following holds for infinitely many n: $C_n(\overline{x}) > c \ln n$, where[2] $c = .120\cdots$.

[1] Means that every interval has the right amount of points 'in the limit'.
[2] To be precise, $c = \max_{x>3} \frac{x-2}{4(x-1)\ln x}$.

A Surprising Result

Theorem. [13] The **clumpiness** of $\{i\rho\}_{i=1}^{n}$ equals the **length of the support** of $\nu(\rho, n, x)$.



The Proof is elementary and consists of three steps.
1: The symmetries on page 12 imply that

$$\nu(\rho, n, x) = \nu(1 - \rho, n, x) = \nu(\rho, n, -x)$$

2: This discontinuities of $S(\rho, n, x)$ are at $\{-i\rho\}$. Re-label these as y_i in ascending order in $[0, 1)$. Then express $\sup S - \min S$ in terms of the y_i.
3: Show that expression obtained equals the clumpiness. **QED**

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THE SHAPE OF

$\nu(\rho, q_n, z)$

Sums of Fractional Parts of $i\rho$

Proposition. [13] Let $\gcd(p,q)=1$ and set $d := q\rho - p$. If $|d| < 1/(q-1)$, then

$$\begin{aligned}\sum_{i=1}^{q}\{i\rho\} &= \frac{(q+1)d+q-1}{2} - \lfloor d \rfloor \\ \sum_{i=1}^{q} \lfloor i\rho \rfloor &= \frac{(q+1)p-q+1}{2} + \lfloor d \rfloor\end{aligned}$$

Idea of Proof. $S(\rho, q, 0)$ can be given in two ways:

(1) $S(\frac{p}{q}, q, 0) = \sum_{i=1}^{q} \left\{ i\frac{p}{q} \right\} = \sum_{i=1}^{q} \left\{ \frac{i}{q} \right\} - 1 - \frac{q}{2}$.

(2) $S(\frac{p}{q}, q, 0) = q \cdot 0 + \frac{q(q+1)}{2}\rho - \frac{q}{2} - \sum_{i=1}^{q} \left\lfloor i\frac{p}{q} \right\rfloor$.

(1) can be computed exactly.
Equate to (2) to get expression for $\sum_{i=1}^{q} \left\lfloor i\frac{p}{q} \right\rfloor$.
But this is equal to $\sum_{i=1}^{q} \lfloor i\rho \rfloor$ for ρ close to p/q.
Compute $\sum_{i=1}^{q} i\rho$.
The difference yields the proposition. **QED**

This result can now be leveraged to write the q branches of

$$S(\rho, q, x) = qx + \frac{q(q+1)\rho - q}{2} - \sum_{i=1}^{q} \lfloor x + i\rho \rfloor$$

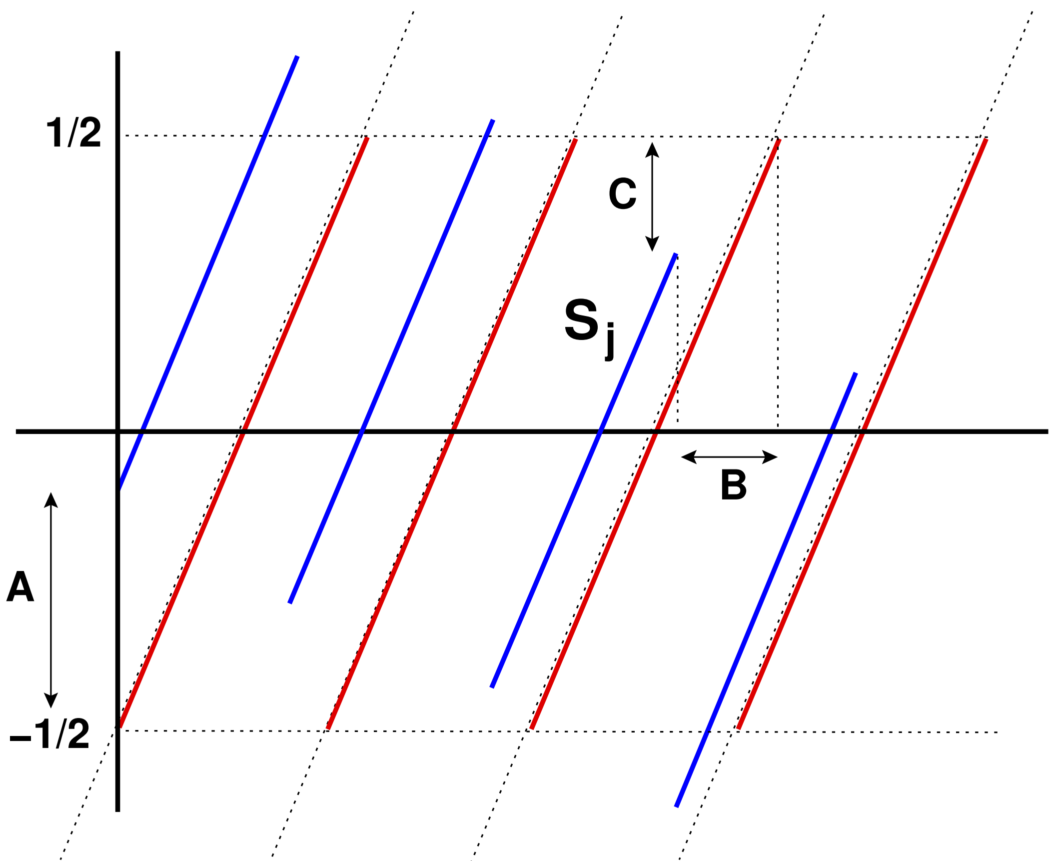
explicitly.

Movement of the Branches

Let p/q a continued fraction denominator of ρ and $d := q\rho - p$. Compute $S((p+td)/q, q, x)$ when d small:

$$qx + \frac{(q+1)(p+td)-q}{2} - \sum_{i=1}^{q} \left\lfloor x + i\frac{p+td}{q} \right\rfloor$$

where t is going from 0 to 1. See Figure. A, B, and C, are, respectively, $\frac{(q+1)d}{2}$, $\frac{i_+ d}{q}$, and $\frac{(q+1-2i_+)d}{2}$.

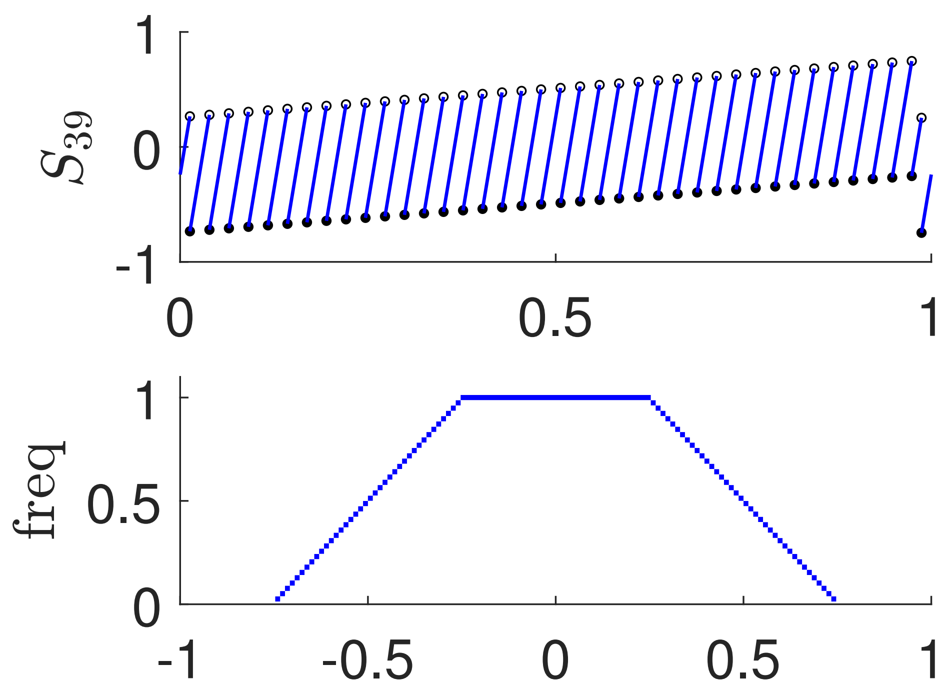


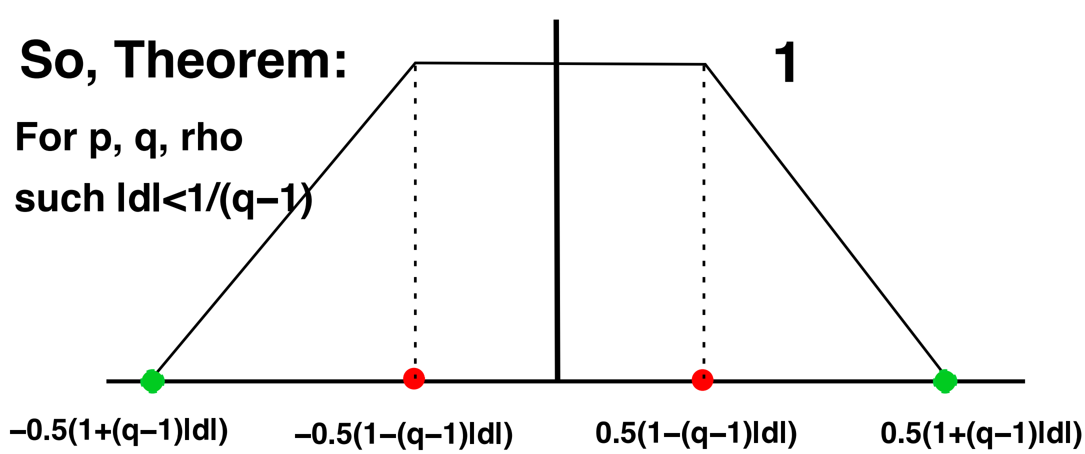
As long as the jth branch satisfies: $S_j(x_{\text{left}}) < 0$ and $S_j(x_{\text{right}}) > 0$, the number of inverse images of 0 equals q, ie: $\nu(0) = 1$.

The Trapezoid Theorem

In analyzing this one sees that the reasoning is completely **independent** of p as long as it is a reduced residue modulo q. So set $\rho' = \rho - \frac{p-1}{q} \approx 1/q$:

$$\nu(\rho, q, z) = \nu(\rho', q, z)$$



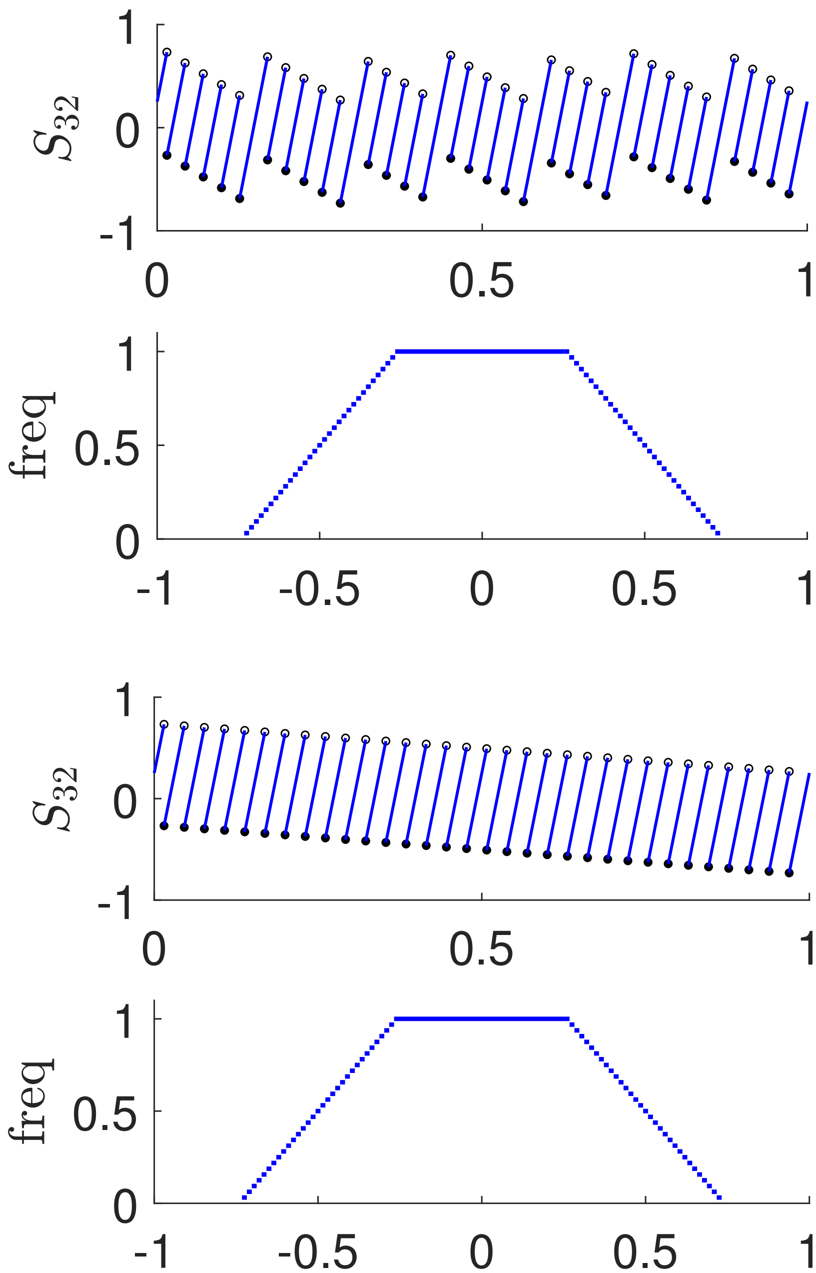


Example of Trapezoid Theorem I

$\frac{23}{32}$ and $\frac{28}{39}$ are successive approximants of $e-2$.

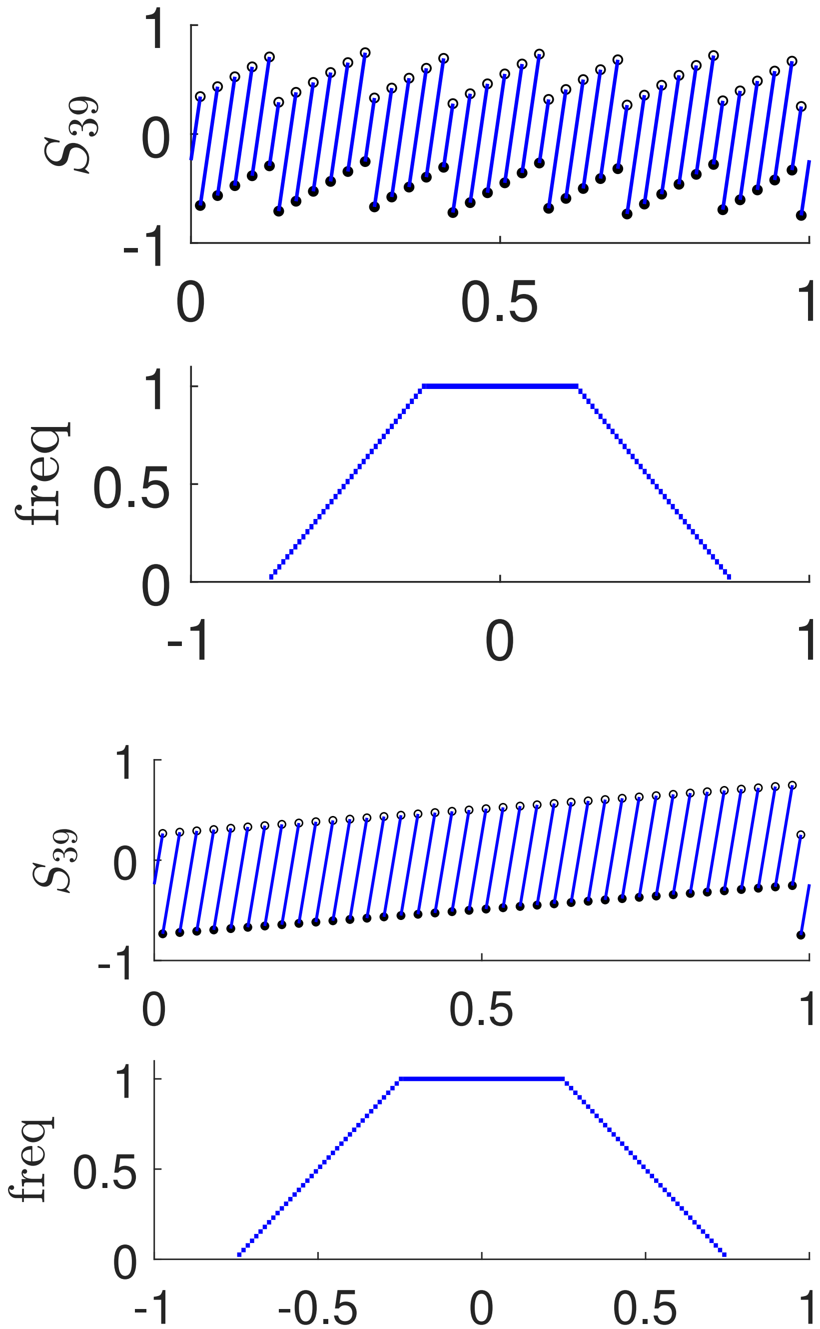
$\boldsymbol{e-2-22/32=[32,2,18,\cdots]}$.

The "2 interval thm" applies [12]: 31 short ones and 1 long.



Example of Trapezoid Theorem II

$\frac{23}{32}$ and $\frac{28}{39}$ are successive approximants of $e - 2$.
$\boldsymbol{e - 2 - 27/39 = [38, 2, 1532, \cdots]}$.
The “3 interval thm” applies [12]: 37 medium, 1 short, 1 long.



Bonus Theorem about Discrepancy

Bonus Theorem. For ρ, p, and q such that

$$|d| = |q\rho - p| < 1/(q-1)$$

the clumpiness (or q times the discrepancy) of $\{i\rho\}_{i=1}^{q}$ equals $1 + (q-1)|d|$ (exactly).

Proof. Because the clumpiness equals the length of the support of ν (Theorem, page 17). **QED**

Idea. Let q_n be the continued fraction denominators of ρ. Figure out how the clumpiness of $\{i\rho\}_{i=1}^{q_n+q_k}$ with $k < n$ affects the length of the support of ν.

Then you may be able to derive precise upper bounds for the clumpiness (ie discrepancy) of $\{i\rho\}_{i=1}^{q_n+q_k}$ as $n \to \infty$. Generalizing that, and writing arbitrary n as sums of q_n, we may get the running max of the clumpiness of $\{i\rho\}_{i=1}^{j}$, for $j \in \{1, \cdots, q_n\}$.

Note. Similar to what we do later with $S(i)$.

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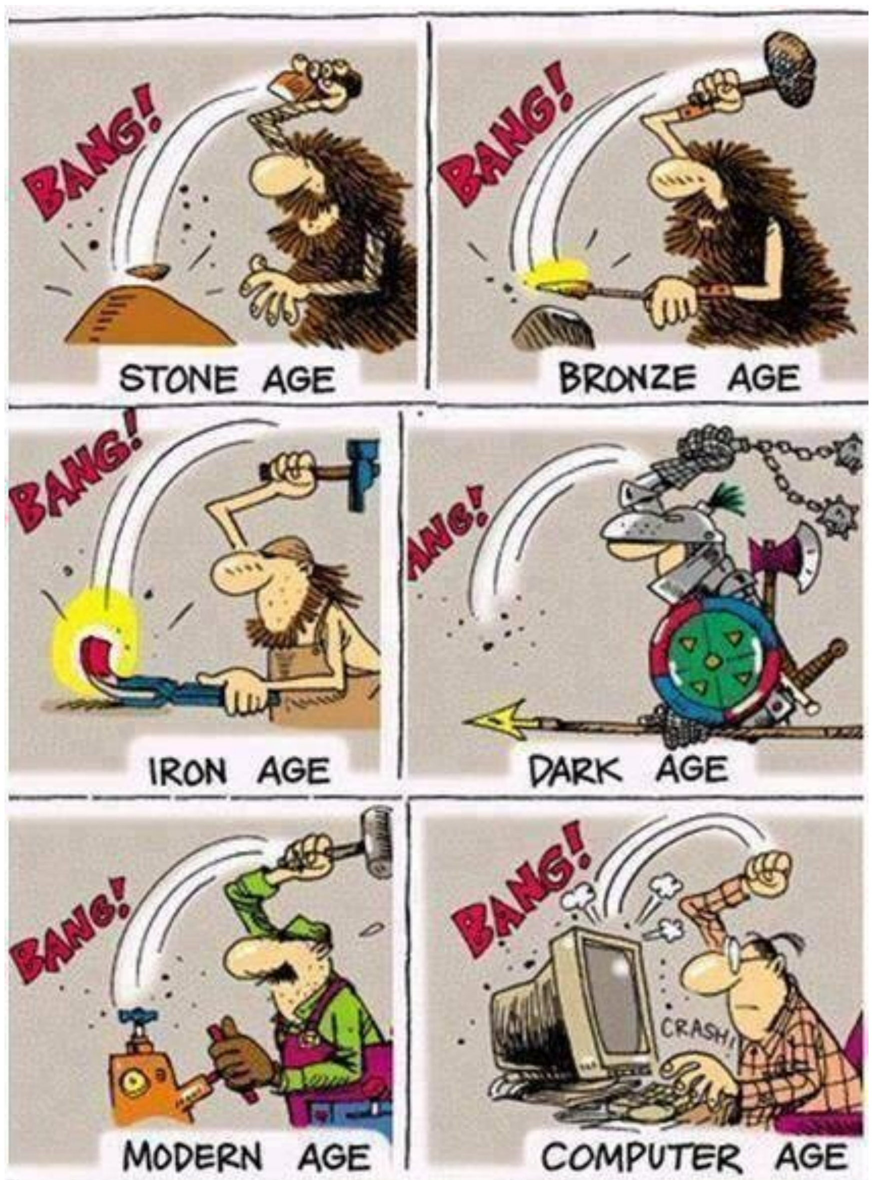
BIRKHOFF SUMS OF THE METALLICA



Key Result I

Definition. Fix a rot. number ρ, set $x = 0$, and abbreviate

$$\boldsymbol{S(n) := S(\rho, n, 0) = \sum_{i=1}^{n} \left(\{i\rho\} - \frac{1}{2} \right)}$$

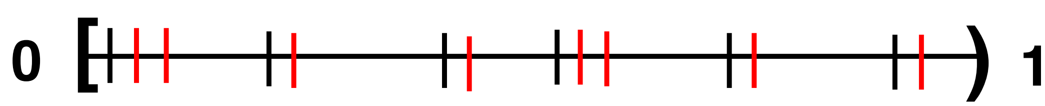
- p_n/q_n are the cont'd fr. approximants of ρ.
- $d_n := q_n\rho - p_n$.

Theorem. $\boldsymbol{S(q_n) = (-1)^n \frac{(q_n + 1)|d_n| - 1}{2}}$

Proof. Subtract $q_n/2$ from Prop. page 19, rework. **QED**

Note. We know $q_n|d_n| < \rho$ [12] (exerc. 6.12). So $S(q_n) > 0$ if n odd, and $S(q_n) < 0$ if n even.

See Figure below.
Black: $\{i\rho\}$ with $i \in \{1, \cdots, q_n\}$.
Then in red: $\{i\rho\}$ with $i \in \{q_n + 1, \cdots, q_n + i\}$.



Red position $=$ black position $\boldsymbol{+ d_n}$.
Sometimes various times over (if $a_{n+1} > 1$).

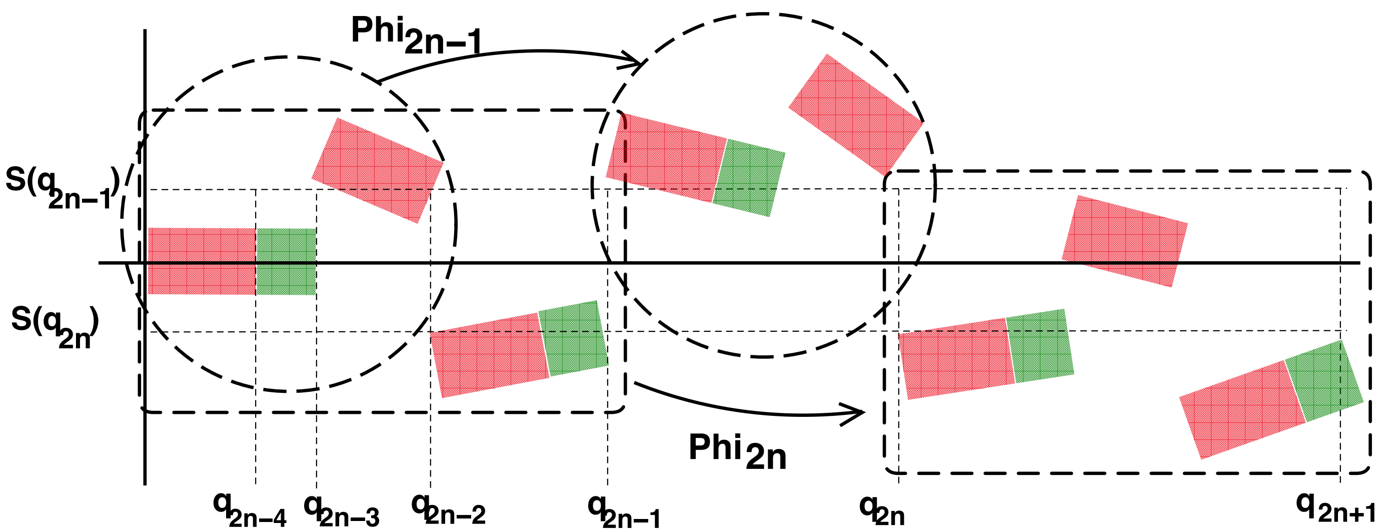
Key Result II

Reasoning like that gives the next key result.

Theorem. For all i with $0 \le i < (a_{n+1} - 1)q_n + q_{n-1}$:

$$S(q_n + i) = S(q_n) + S(i) + i d_n \,.$$

The series $S(i)$ is "self-similar" by affine maps.
Below we sketch the situation for the golden mean.



The Metallic Means

Definition. The **metallic means** are (for $a \in \mathbb{N}$)

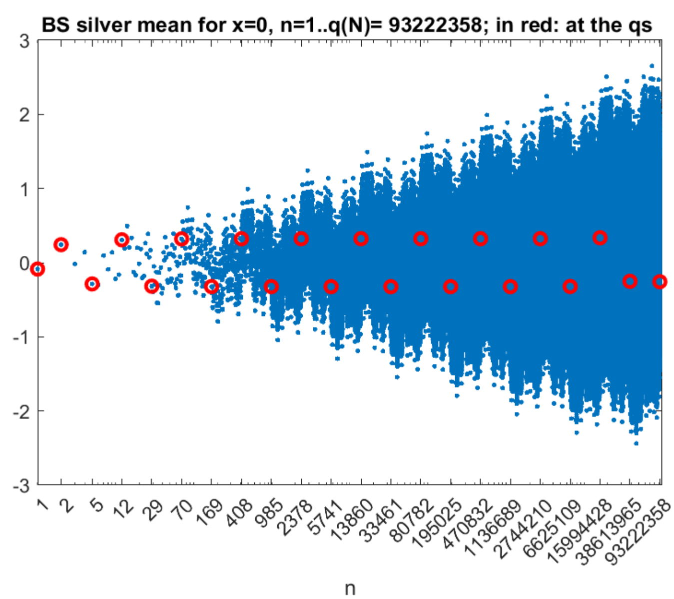
$$\rho_a := [a, a, a, \cdots] = \sqrt{\frac{a^2}{4} + 1} - \frac{a}{2}$$

For a equal to 1, 2, and 3: 'golden', 'silver', and 'bronze'.

Lemma. For the metallic mean $\rho_a := [a, a, a, \cdots]$, we have

$$d_n = (-1)^n \rho_a^{n+1} \qquad \text{and} \qquad q_n = \frac{\rho_a^{-n-1} - (-\rho_a)^{n+1}}{\sqrt{a^2 + 4}}$$

The plan: find $\boldsymbol{M_n}$: the **max** of S on $\{0, 1 \cdots, q_{2n+1}\}$. Then find $\boldsymbol{m_n}$: the **min** of S on $\{0, 1 \cdots, q_{2n+2}\}$.



This appears to determine constants $K_a > 0$ so that $S(M_n) - K_a n$ and $S(m_n) + K_a n$ are bounded.

A Conjecture and Numerical Evidence

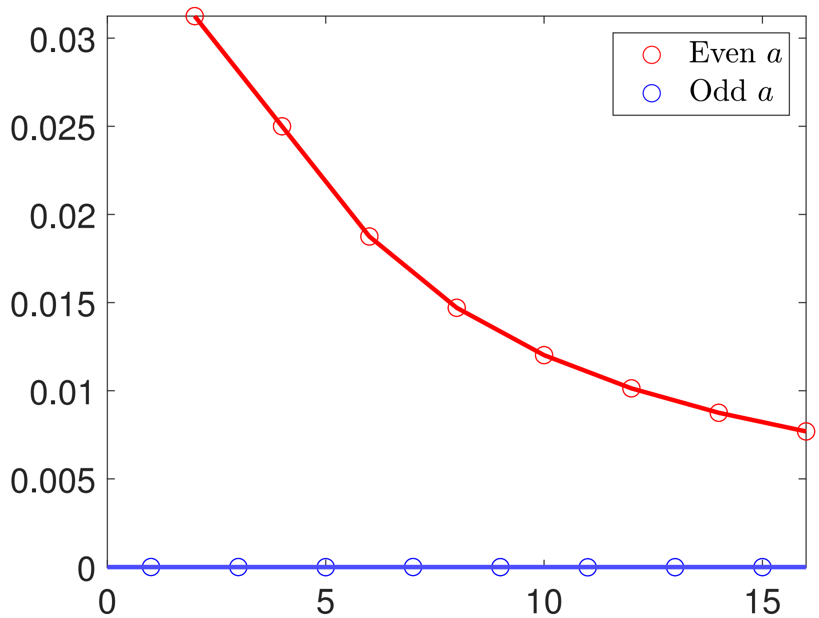
Recall: $\boldsymbol{M_n}$ is the **max** of S on $\{0, 1 \cdots, q_{2n+1}\}$, while $\boldsymbol{m_n}$ is the **min** of S on $\{0, 1 \cdots, q_{2n+2}\}$.

Conjecture. $|S(M_n) - K_a n|$ and $|S(m_n) + K_a n|$ are bounded

$$\text{where} \qquad K_a = \begin{cases} \dfrac{a}{8} & a \text{ even} \\ \dfrac{a(a^2+3)}{8(a^2+4)} & a \text{ odd} \end{cases}$$

Numerical Evidence. Compute K_a for $a \in \{1, 2, \cdots, 16\}$. Check accuracy of computation: know $\boldsymbol{S(q_n)}$ **exactly**.

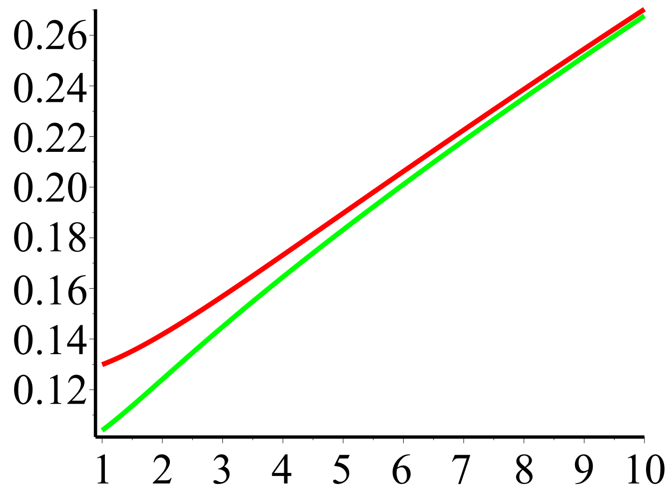
Plot $\quad \boldsymbol{K_a - \dfrac{a(a^2+3)}{8(a^2+4)}}$. See below.



Remarks

Observation: Since $\ln M_n = c_a + 2n \ln \rho$ where c_a is bounded, we can compute the following.

Corollary to Conjecture: $\limsup_i \dfrac{S(i)}{\ln i} = \zeta(a)$ and $\zeta(a)$ is plotted for even a is red, and odd a in green.



More specifically, for $a \in \{1, \cdots, 8\}$, $\zeta(a)$ equals:

0.1039043458
0.1418240820
0.1448629492
0.1731739099
0.1831704913
0.2062199841
0.2183653720
0.2386962361

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PROOF OF THE CONJECTURE FOR THE SILVER MEAN

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Data for the Silver Mean

We have proof of the conjecture for the golden and silver means and partial results for some other metallica. We outline the proof for the silver mean.

The silver mean ρ_2 equals $\sqrt{2}-1$. Some convergents, starting with $\frac{p_0}{q_0}$

$$0,\ \frac{1}{2},\ \frac{2}{5},\ \frac{5}{12},\ \frac{12}{29},\ \frac{29}{70},\ \frac{70}{169},\ \frac{169}{408},\ \frac{408}{985},\ \frac{985}{2378},\ \frac{2378}{5741}$$

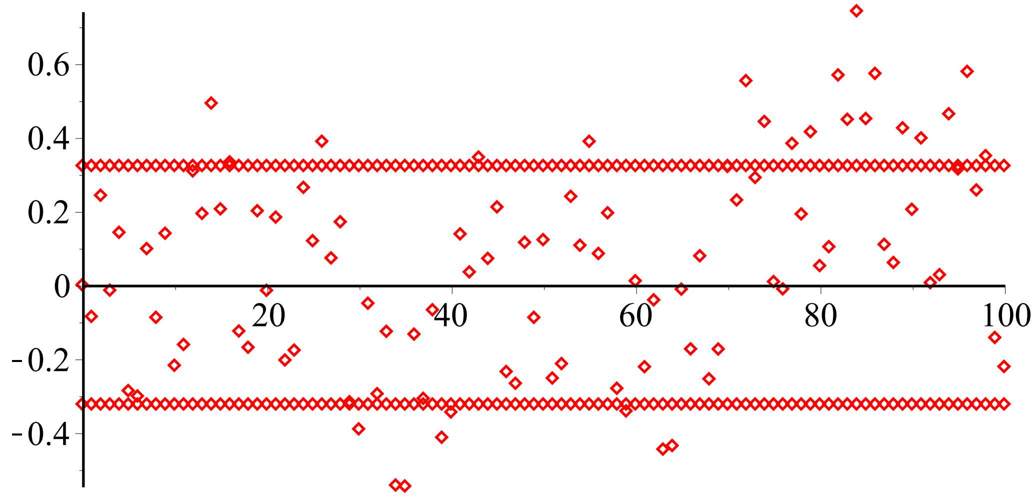
Define M_n and m_n as $\begin{cases} M_n &= \sum_{i=0}^{n-1} q_{2i+1} \\ m_n &= \sum_{i=0}^{n} q_{2i} \end{cases}$. We list the first few, starting with M_0, and m_0:

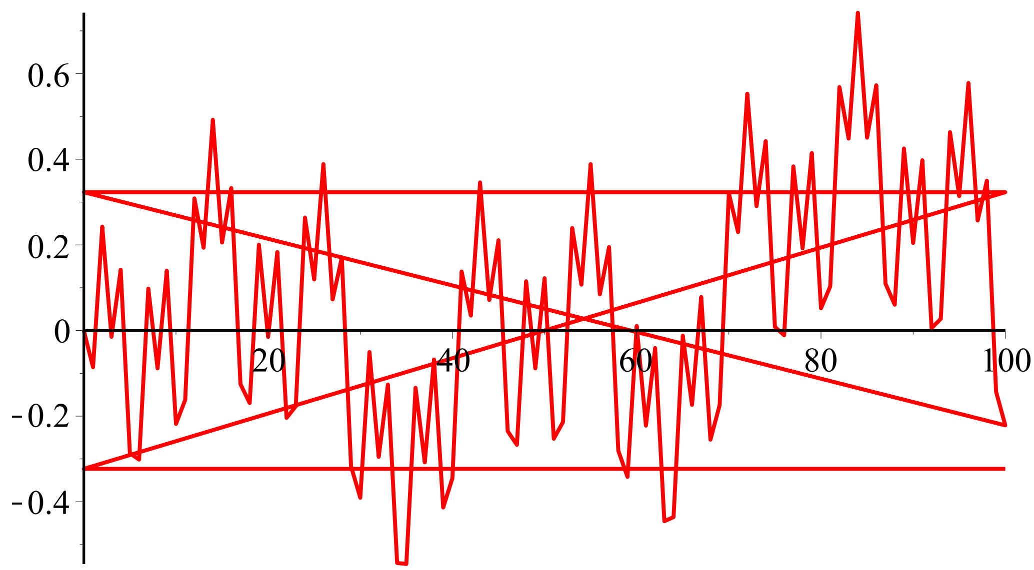
$$M_n = \ 0,\ 2,\ 14,\ 84,\ 492,\ 2870$$
$$m_n = \ 1,\ 6,\ 35,\ 204,\ 1189,\ 6930$$

Recall that

$$d_n = (-1)^n \rho_2^{n+1} \qquad \text{and} \qquad q_n = \frac{\rho_2^{-n-1} - (-\rho_2)^{n+1}}{\sqrt{2^2+4}}$$

$S(i)$ for the Silver Mean





$$\frac{p_n}{q_n} = \ 0,\ \frac{1}{2},\ \frac{2}{5},\ \frac{5}{12},\ \frac{12}{29},\ \frac{29}{70},\ \frac{70}{169},\ \frac{169}{408},\ \frac{408}{985},\cdots$$

$$M_n = \ 0,\ 2,\ 14,\ 84,\ 492,\ 2870,\cdots$$
$$m_n = \ 1,\ 6,\ 35,\ 204,\ 1189,\ 6930,\cdots$$

First Part of the Proof

Proposition I. For M_n as defined, we have

$$S(M_{n+1}) - S(M_n) = \frac{1}{4}(1 - \rho^{4n+4})$$

Idea of Proof. Since $M_{n+1} = q_{2n+1} + M_n$, use Key Result II (pg 27) to see

$$\begin{aligned} S(M_{n+1}) &= S(q_{2n+1}) + S(M_n) + M_n d_{2n+1} \\ \text{So} \quad S(M_{n+1}) - S(q_{2n+1}) &= S(M_n) + M_n d_{2n+1} \end{aligned}$$

Now use Lemma pg 29 to compute $S(q_n)$ and $M_n d_{2n+1}$ in terms of ρ. **QED**

Warning. This sounds obvious, but remember you do not a priori know what the values M_n are.

Note. For the minima similar estimates work, because their definition is essentially the same.

Corollary. In fact,

$$S(M_n) - \frac{n}{4} = \frac{-\rho^4}{4(1-\rho^4)} + O(\rho^{4n})$$

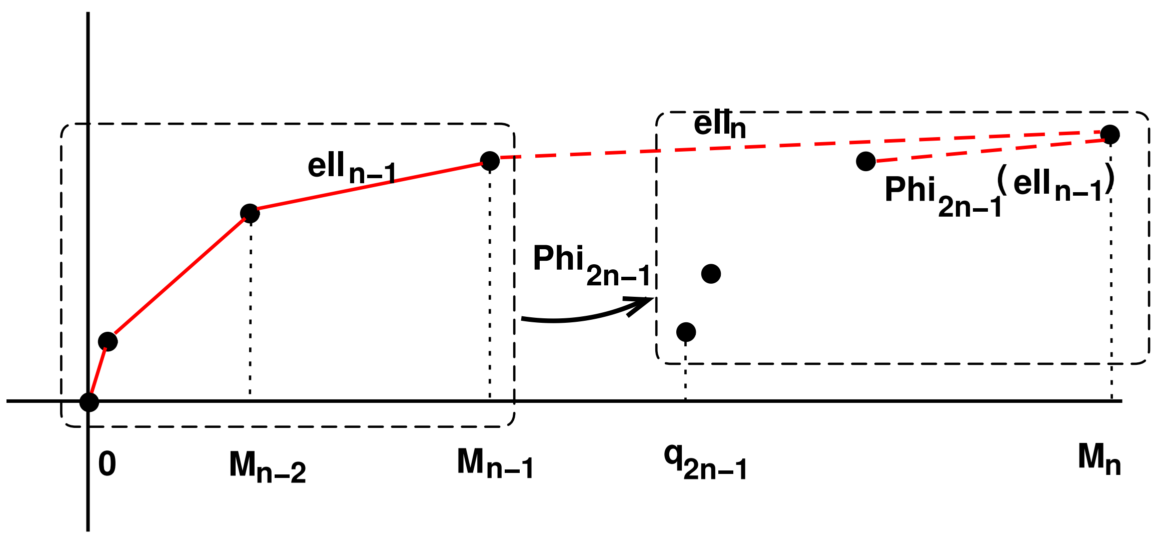
Second Part of the Proof

Proposition II. $S(M_n) > S(i)$ for all i in $\{0, 1, 2, \cdots, q_{2n+1} - 1\}$ (except when $i = M_n$).

Main Steps of Proof. Step 1. Define M_i as in part I. Use Key Result II to show by induction that

$$S(\rho, M_n, 0) > S(\rho, i, 0) \quad \text{for all} \quad 0 \leq i < q_{2n-1}\,.$$

Step 2. Connect the points $(M_{i-1}, S(M_{i-1}))$ and $(M_i, S(M_i))$ by a segment ℓ_i (see figure). Show that the image under Φ_{2n-1} of ℓ_{n-1} is increasing for all n. **QED**



Note. Again, the computation is the same for the minima.

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APPENDIX

MORE PICTURES

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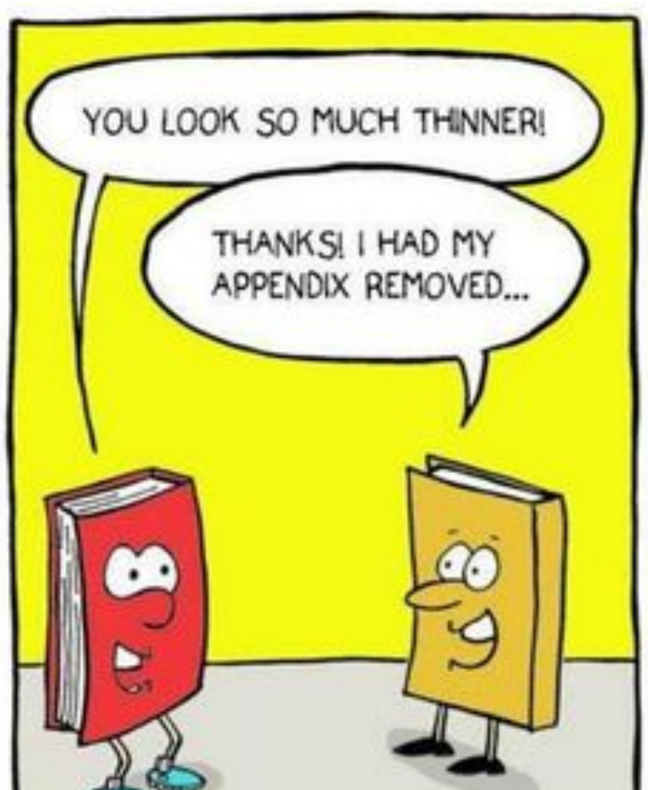
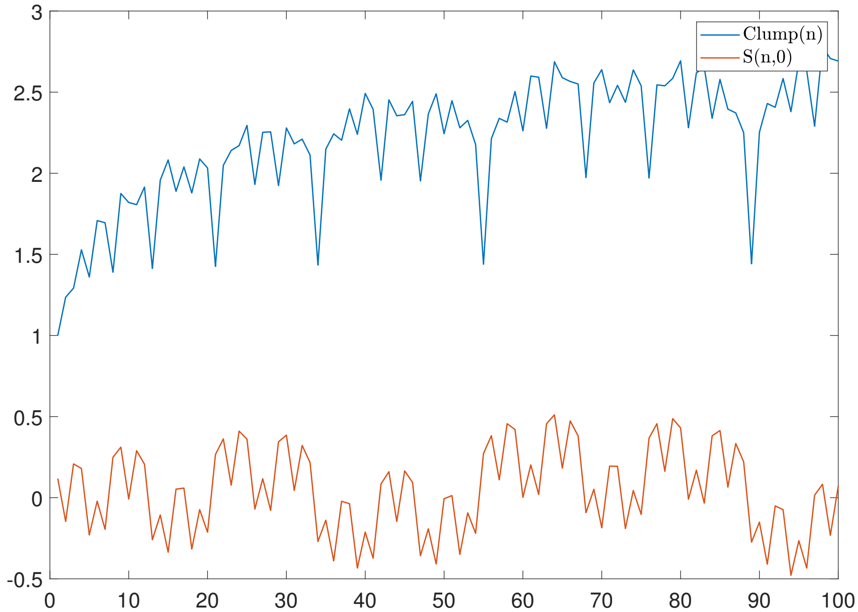
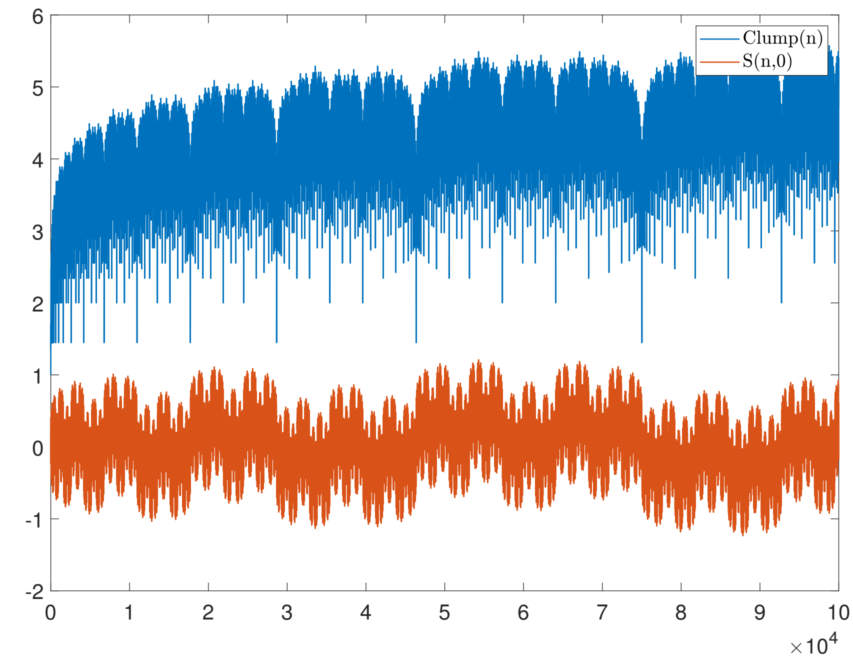


The Golden Mean

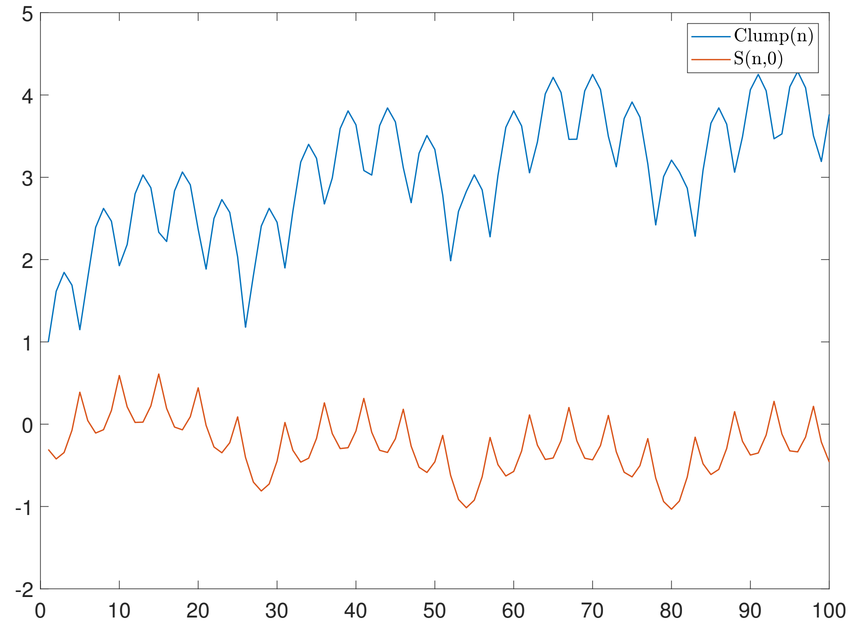
First $i \in \{1, \cdots 100\}$, then $i \in \{1, \cdots 10^5\}$.

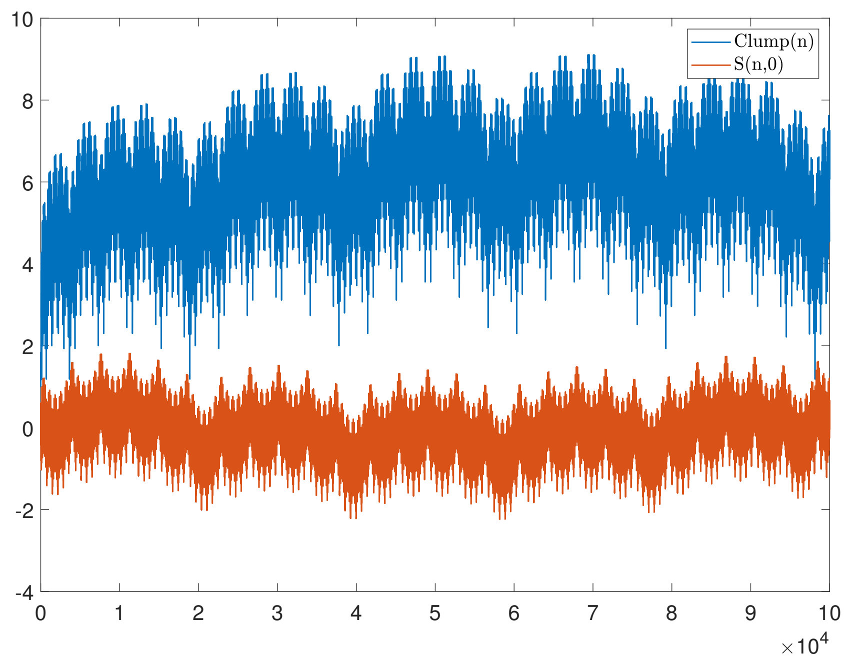




The Metallic-5 Mean

First $i \in \{1, \cdots 100\}$, then $i \in \{1, \cdots 10^5\}$.





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