11 Derivatives, Again, Measure Theoretically

The first section recalls the classical definition and the second recalls derivatives as linear approximation as a launching point to a very geometric measure theoretical way of looking at derivatives. The remaining sections significantly expand our previous explorations using measure theoretic tools.

11.1 secants and derivatives

The derivative that is encountered for the first time in calculus is defined as the limit of a ratio of the "rise" over "run" of the graph of a function. For $y = f(x)$, this becomes

$$\frac{df}{dx}(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

This is visualized as the slope of the secant lines approaching a limit – the slope of the tangent line – as the free ends of those lines approach $(a, f(a))$. Figure 66 illustrates this.

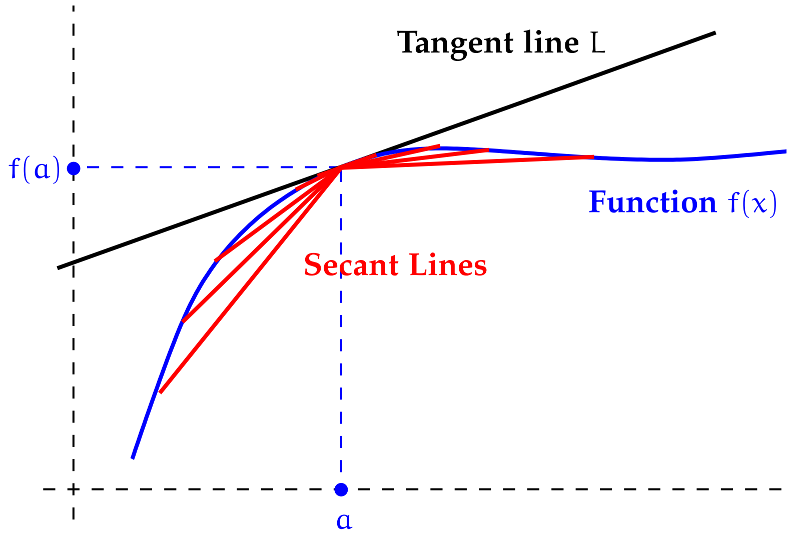

Figure 66: The traditional definition of the derivative

11.2 THE DERIVATIVE AS AN APPROXIMATION

The derivative as $\hat{L}_a$, the optimal linear approximation to f at a, is another, very useful way to think about the derivative. Here, we focus on the fact that the tangent line at $(a, f(a))$ approximates the graph of $f(x)$ at $(a, f(a))$ as we zoom in on the graph. More precisely, writing $x = h + a$,

$$f(x) = f(h + a) = f(a) + \hat{L}_a(h) + g(h)h,$$

where $\hat{L}_a$ is linear in h, $g(h) \to 0$ as $h \to 0$, and the tangent line L is the graph of the function $y = f(a) + \hat{L}_a(x - a)$.

Exercise 11.2.1. Use the facts that (1) linear $\hat{L}_a : \mathbb{R} \to \mathbb{R}$ have the form $h \to sh$, s a scalar, and (2) $g(h) \to 0$ as $h \to 0$, to rearrange this last equation for $f(x)$ into the original definition of a derivative.

Using the equation above to get

$$\left| f(x) - \left(f(a) + \hat{L}_a(x - a) \right) \right| \leqslant \left(\sup_{|s| \in [0, \epsilon]} |g(s)| \right) |h| \text{ for } h \in [-\epsilon, \epsilon],$$

which has the nice geometric interpretation illustrated in Figure 67. The figure illustrates the fact that the graph of $f(x)$ lies in cones

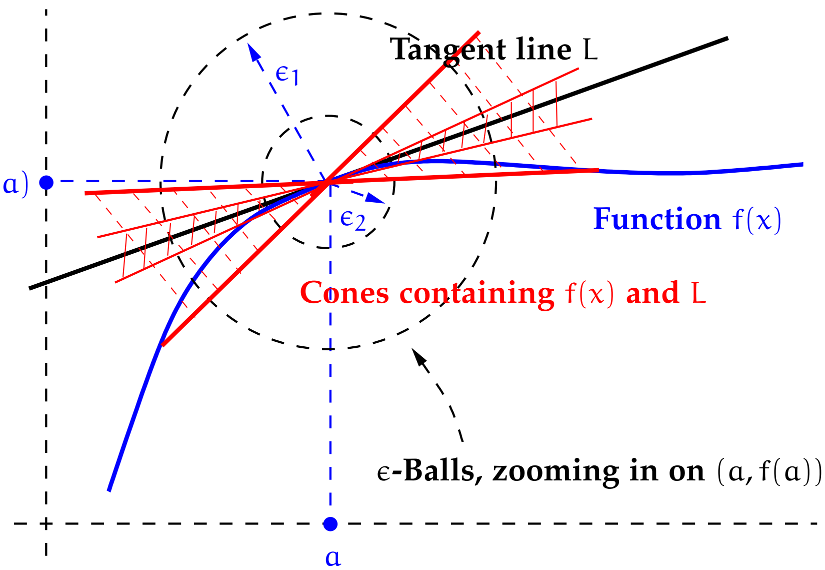

Figure 67: As we zoom into a point of differentiability, the graph is contained in cones that get thinner.

centered on L, whose angular widths go to zero as we restrict ourselves to smaller and smaller ϵ-balls centered on $(a, f(a))$. Inside the ϵ_1-ball, the graph stays in the wider cone, while in the smaller, ϵ_2-ball the graph stays in the narrower cone.

Let's restate this. Defining

- $p \equiv (a, f(a))$,
- $B(\epsilon)$ to be the ball of radius ϵ centered on p,
- $F \equiv \{(x, y)|y = f(x)\}$,
- $C_L(p, \epsilon)$ to be the smallest closed cone, symmetrically centered on L, with vertex at p such that $F \cap B(\epsilon) \subset C_L(p, \epsilon)$, and
- $\theta(\epsilon)$ to be the angular width of $C_L(p, \epsilon)$,

we have that f is differentiable at $a \Leftrightarrow \theta(\epsilon) \to 0$ as $\epsilon \to 0$. Figure 68 illustrates this.

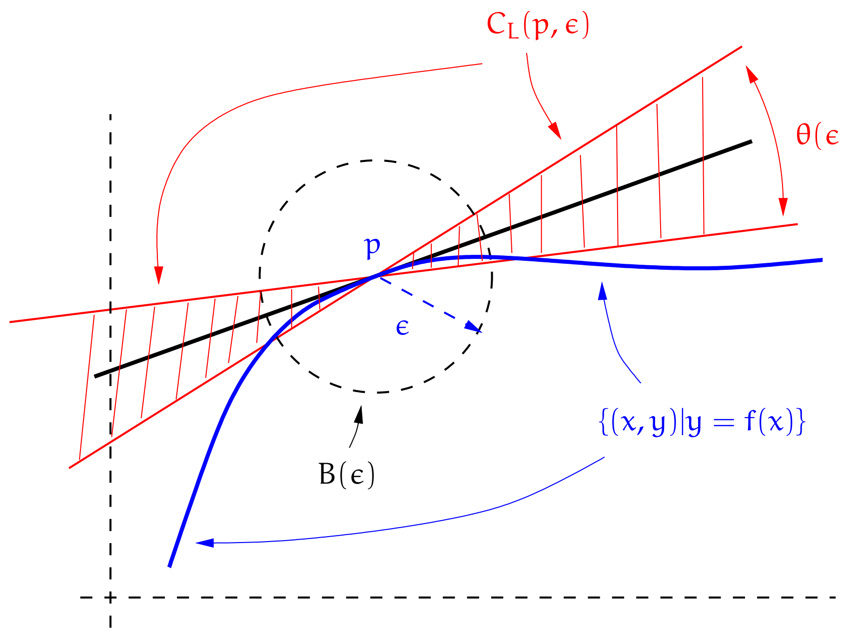

Figure 68: f is differentiable at $a \Leftrightarrow \theta(\epsilon) \to 0$ as $\epsilon \to 0$

Exercise 11.2.2. Provide the missing details taking us from the above inequality bounding the deviation from linearity to the above statement that $\{$f is differentiable at $a \Leftrightarrow \theta(\epsilon) \to 0$ as $\epsilon \to 0\}$ using the facts that (1) the above inequality defines cones that are almost symmetric about L and (2) the ϵ-ball centered at p is contained in the vertical strip $(x - a, y - f(a)) \in [-\epsilon, \epsilon] \times (-\infty, \infty)$

With this shift to a geometric perspective, we are now in a position to take a step in the direction of geometric measure theory. Note that in our definition the cones contain all of the graph as they narrow down and we zoom in. What if instead of the cones converging to a line, we converge to two or three lines, or that we converge to a cone that does not narrow, or two cones that do not narrow? Then we are interested in the general tangent cone, a special case of which is the usual tangent line. Alternatively, what if all we know is that a larger and larger fraction of the graph is in a narrower and narrower cone as we zoom into p? That is precisely the idea that approximate tangent lines capture. We now turn to the first idea, the tangent cone.

11.3 TANGENT CONES

The tangent line discussed above is also the tangent cone. The tangent cone of a set in $\mathbb{R}^n$ can have any dimension from 1 to n. For nicely behaved k-dimensional sets, the tangent cone will also be k-dimensional. In the case of the usual derivative of functions from $\mathbb{R}$ to $\mathbb{R}$, we are working in the graph space $\mathbb{R}^2$ with 1-dimensional sets. Moving to tangent cones, we can approximate one dimensional sets which are not graphs or, more generally, arbitrary subsets of $\mathbb{R}^n$.

We now build up to a definition of the tangent cone of $F \subset \mathbb{R}^n$ at p. Begin by translating F by $-p$. (This moves p to 0.) Define $F(\epsilon) \equiv (F \cap B(\epsilon)) \setminus p$. Use a projection center at 0 to project the translated $F(\epsilon)$ onto the sphere of radius ϵ Take the closure of the resulting subset of the ϵ-sphere. Finally take the cone over this set. Call this set $T_p^\epsilon(F)$ (We will sometimes refer to this as the tangent cone at scale ϵ). Putting all this together,

$$T_p^\epsilon(F) = \{\mathbb{R} \geqslant 0\}(\text{Closure}(\cup_{x \in F(\epsilon)} \frac{x-p}{|x-p|})).$$

Now the tangent cone of F at p is the intersection of $T_p^\epsilon(F)$ at any sequence of ϵ_i's going to zero; $\epsilon_i = \frac{1}{i}$ will do. Thus the tangent cone of F at p, $T_p(F)$ is given by:

$$T_p(F) = \bigcap_i T_p^{\frac{1}{i}}(F).$$

Summarizing, we get:

Definition 11.3.1. *The tangent cone of* F *at* p *is given by*

$$T_p(F) = \bigcap_i T_p^{\frac{1}{i}}(F).$$

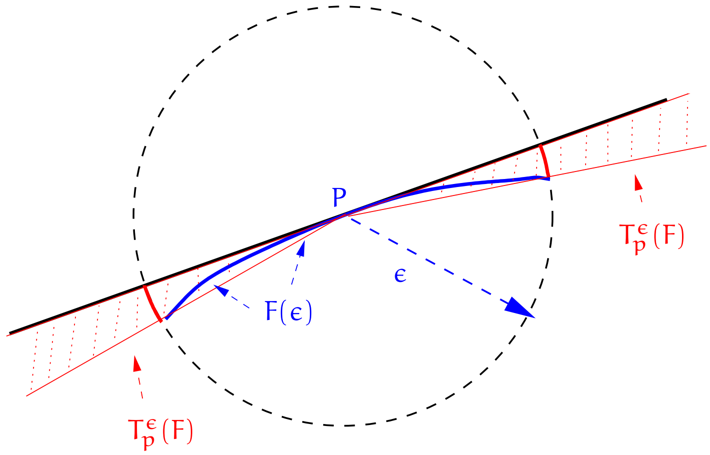

Figure 69: As we zoom in, the tangent cone at scale ϵ, $T_p^\epsilon(F)$, converges to the tangent line through p.

where $T_p^\epsilon(F)$ *is given by*

$$T_p^\epsilon(F) = \{\mathbb{R} \geqslant 0\}(Closure(\cup_{x \in F(\epsilon)} \frac{x-p}{|x-p|})).$$

In the case of a differentiable function $f : \mathbb{R} \to \mathbb{R}$, this tangent cone is the usual 1-dimensional tangent line. Figure 69 illustrates this.

Remark 11.3.1. *The tangent cone is centered on the origin, 0, but I will be plotting it as though it were centered on* p. *Similarly, the tangent lines will sometimes be thought of as linear subspaces (i.e. centered on the origin 0, and other times as the shift of that linear subspace to* p.

If the curve we are considering does not have a derivative at p, then we can get tangent cones that are not lines. Figure 70 illustrates an example of this. The function generating the cone in the figure keeps oscillating between the upper and lower lines as we zoom into p.

Exercise 11.3.1. Construct an concrete example of a function that has a tangent cone like the tangent cone shown in Figure 70.

Exercise 11.3.2. Come up with examples of one dimensional sets in $\mathbb{R}^2$ which have a tangent cone at p equaling:

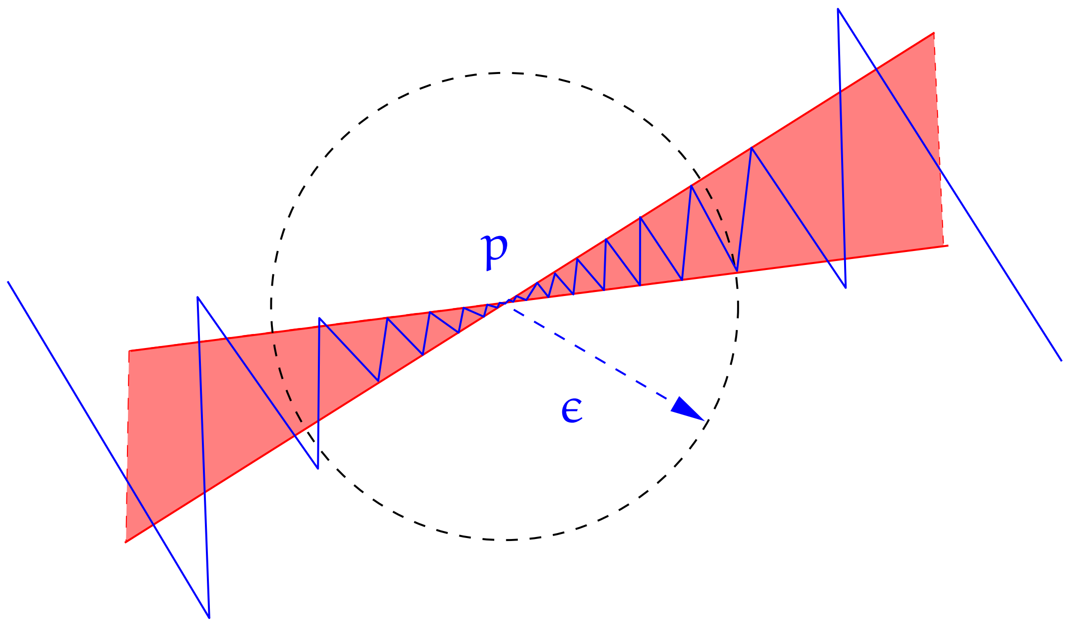

Figure 70: The tangent cone of the blue curve at the point p is shown in red and does not narrow down to a line as $\epsilon \to 0$.

- Two lines passing through p,
- An infinite number of lines passing through p, yet **not** equaling all of $\mathbb{R}^2$
- one convex cone with vertex at p.
- a line through p and one convex cone with vertex at p.

Can you do this using only 0 dimensional sets?

11.4 APPROXIMATE TANGENT CONES

11.4.1 *Densities*

Now we need $\theta^k(\mu, F)$, the k-dimensional density of F at p. Define $\alpha(k)$ to be the volume of the unit ball in $\mathbb{R}^k$ when k is an integer, and something that interpolates sensibly otherwise. (There is a standard way to do this using Γ functions). Choose some measure μ. (Typically this will be k-dimensional Hausdorff measure, $\mathcal{H}^k$, restricted to some set, possibly with some weight function). Now, $\theta^k(\mu, F)$ is given by

$$\theta^k(\mu, F) = \lim_{\epsilon \to 0} \frac{\mu(F \cap B(\epsilon))}{\alpha(k)\epsilon^k}$$

when this limit exists. When the limit does not exist, we work with the limsup and liminf of the right hand side which are called upper and lower densities of F at p and are denoted by $\theta^{*k}(\mu, F)$ and $\theta^k_*(\mu, F)$ respectively.

11.4.2 *Using Densities to get Approximate Tangents*

We now define the approximate tangent cone at p to be the intersection of closed cones whose complements intersected with F have density zero at p:

$$\tilde{T}_p(F) = \bigcap \{\text{closed cones } C \text{ with vertex } p | \theta^k(\mu, (\mathbb{R}^n \setminus C) \cap F) = 0\}$$

Originally (in this section), we were aiming at having a definition of approximate tangent line that was invariant to (small) pieces of the set F outside the sequence of cones, provided those pieces got small enough, quick enough. Now we can make that more precise. We want a definition of approximate tangent line that ignores such excursions of F provided these excursions have density zero at p.

Rather anti-climatically then, here is the definition we have been waiting for (though you might have already guessed it!). A 1-dimensional set has an *approximate tangent line* at p when the approximate tangent cone is equal to a line through p. When the 1-dimensional set is an embedded differentiable curve, the tangent line and the approximate tangent line are the same.

Remark 11.4.1. *In general, when we are dealing with* k*-dimensional sets in* $\mathbb{R}^n$*, we will get approximate tangent* k*-planes. That is because most things we deal with will be* rectifiable sets *having approximate tangent* k*-planes* $\mathcal{H}^k$ *almost everywhere. Rectifiable sets are introduced in Chapter 18.*

Exercise 11.4.1. Can you create examples of one dimensional sets which have a (density based) approximate tangent line at p but not the usual tangent line at p?

Exercise 11.4.2. Prove that a tangent line to a continuous curve is also the (density based) approximate tangent line at p.

11.5 WEAK TANGENTS

There is different version of approximate tangent k-plane based on integration. We will call these tangents, *weak tangents*.

We start with the fact that we can integrate functions defined on $\mathbb{R}^n$ over k-dimensional sets using k-dimensional measures μ (typically $\mathcal{H}^k$). We zoom in on the point p, through dilation of the set F:

$$F_\rho(p) = \{x \in \mathbb{R}^n | \ x = \frac{y-p}{\rho} + \ p \text{ for some } y \in F\}.$$

We will say that the set F has a *weak tangent k-plane* L at p if the dilation of $F_\rho(p)$, converges weakly to L: i.e. if

$$\int_{F_\rho} \phi \, d\mu \to_{\rho \to 0} \int_L \phi \, d\mu$$

for all continuously differentiable, compactly supported $\phi : \mathbb{R}^n \to \mathbb{R}$.

In Figure 71, we illustrate this for the case of 1-planes - i.e.lines: in the top illustration, L is the weak limit of the dilations of F, while in the bottom it is not.

Exercise 11.5.1. Can you create an example of a one dimensional curve which has the usual tangent line at p but not an (integration based) weak tangent line at p?

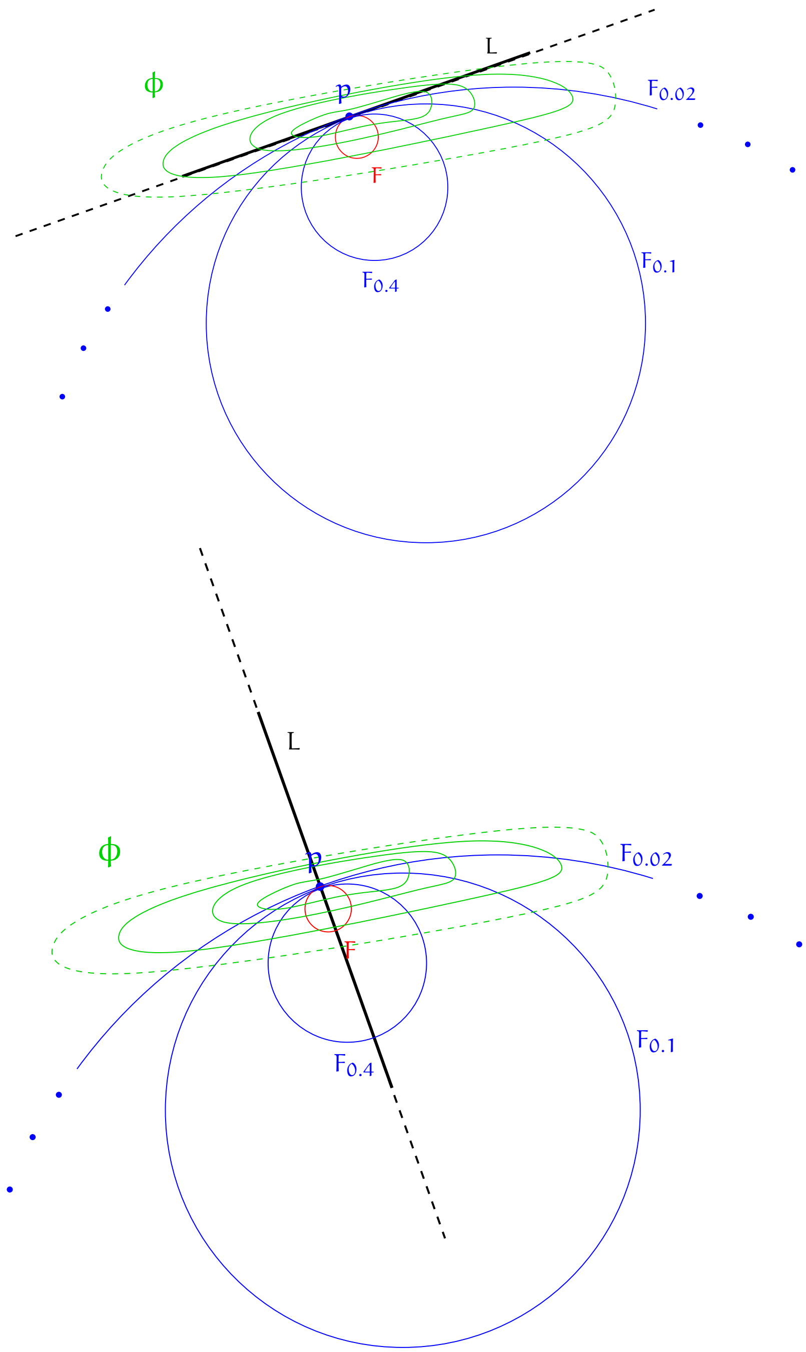

Figure 71: Illustration of a line that is (top) and a line that is not (bottom) the weak tangent. The solid green lines are the level sets of ϕ while the dashed green line indicates the boundary of the support of ϕ. Note also that the ρ's of 0.4, 0.1, and 0.02 are approximate.

11.6 MORE EXERCISES

Exercise 11.6.1. Let E be the set of all points in the unit square $[0,1] \times [0,1]$ which have rational coordinates.

- Find all possible tangent cones generated by E.
- Find all possible approximate tangent cones generated by E and the measure $\mu = \mathcal{L}^2$ (2-dimensional Lebesgue measure).

The idea here is that you can choose the point p at which you are computing the tangent cone anywhere in the plane. Of course if you are not on the closure of E, then the tangent cone is empty (why?) and so you need only consider points in the closure of E.

Exercise 11.6.2. Suppose E is a spiral that spirals around an infinite number of times as it spirals into the origin in $\mathbb{R}^2$.

- What is the tangent cone at $p = (0,0)$?
- What is the approximate tangent cone at $p = (0,0)$ using the measure $\mu = \mathcal{L}^2$?
- What is the approximate tangent cone at $p = (0,0)$ using the measure $\mu = \mathcal{H}^1$ (1-dimensional Hausdorff Measure)?
- What can you say about the tangent cones and approximate tangent cones at all other $p \neq (0,0)$?