On the Toeplitzness of Composition Operators

(with Fedor Nazarov)

1. The setting

- $\mathbb{U} = \{|z| < 1\} \subset \mathbb{C}$

- $H^2 = \{f(z) = \sum_0^\infty \hat{f}(n)\, z^n : \sum |\hat{f}(n)|^2 < \infty\}$

- $H^2 = \{f \in L^2(\partial\mathbb{U}) : \hat{f}(n) = 0 \ \forall n < 0\}$

- $P : L^2(\partial\mathbb{U}) \to H^2$ Orthog (Riesz) proj'n.

2. The operators

- *Composition:* $\varphi : \mathbb{U} \to \mathbb{U}$ holomorphic

  $$C_\varphi f = f \circ \varphi \qquad (f \in \mathrm{Hol}\,(\mathbb{U}))$$

  Littlewood (1925): $\quad C_\varphi(H^2) \subset H^2$

  $C_\varphi : H^2 \to H^2$ *bounded linear operator*

- *Toeplitz:* $b \in L^\infty(\partial\mathbb{U})$

  $T_b f = P(bf) \qquad (f \in H^2)$

  $T_b : H^2 \to H^2$ *bounded linear operator*

- Two Toeplitz Examples: $\quad b(\zeta) = \zeta$
  - $T_b =$ *forward shift* $:= S$
  - $T_{\overline{b}} =$ *backward shift* $= S^*$

3. Matrix of T_b: $\quad b \sim \sum_{-\infty}^{\infty} \widehat{b}(n) e^{in\theta}$

$$[T_b] = \begin{bmatrix} \widehat{b}(0) & \widehat{b}(-1) & \widehat{b}(-2) & \widehat{b}(-3) & \cdots \\ \widehat{b}(1) & \widehat{b}(0) & \widehat{b}(-1) & \widehat{b}(-2) & \cdots \\ \widehat{b}(2) & \widehat{b}(1) & \widehat{b}(0) & \widehat{b}(-1) & \cdots \\ \widehat{b}(3) & \widehat{b}(2) & \widehat{b}(1) & \widehat{b}(0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Constant on Diagonals!

$$S^* \, T_b \, S = T_b$$

4. Which C_φ are Toeplitz?

$$[C_\varphi] = \begin{bmatrix} 1 & \hat{\varphi}(0) & \widehat{\varphi^2}(0) & \widehat{\varphi^3}(0) & \cdots \\ 0 & \hat{\varphi}(1) & \widehat{\varphi^2}(1) & \widehat{\varphi^3}(1) & \cdots \\ 0 & \hat{\varphi}(2) & \widehat{\varphi^2}(2) & \widehat{\varphi^3}(2) & \cdots \\ 0 & \hat{\varphi}(3) & \widehat{\varphi^2}(3) & \widehat{\varphi^3}(3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

- Main diag: $\hat{\varphi}(1) = 1$
- Sub-diag's: $\hat{\varphi}(n) = 0 \quad \forall\, n > 1$
- 1st Super-diag: $\hat{\varphi}(0) = \widehat{\varphi^2}(1) = 2\hat{\varphi}(0)\hat{\varphi}(1)$

Conclude: C_φ Toeplitz $\Leftrightarrow \varphi(z) \equiv z \quad (C_\varphi = I)$

5. Asymptotic Toeplitz Operators
(Barría & Halmos, 1982)

$T \in \mathcal{L}(H^2)$ "Asymptotically Toeplitz"

means:

"$\{S^{*n}\, T\, S^n\}$ strongly convergent."

- "Asymp Toep." $\Rightarrow$ "Diagonals converge"

$$
\begin{aligned}
< S^{*n} T S^n z^\alpha, z^\beta > &= < T S^n z^\alpha, S^n z^\beta > \\
&= < T z^{\alpha+n}, z^{\beta+n} > \\
&= [T]_{\alpha+n,\beta+n}
\end{aligned}
$$

- "Compact" $\Rightarrow$ "Asymp. Toep."

6. Which C_φ are Asymp. Toeplitz?

Theorem 1. $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$

$$\Rightarrow$$

C_φ Asymp. Toep.

Example. $C_{\frac{1+z}{2}}$ Asymp. Toeplitz

nontrivially (not compact).

Proof of Thm. For $f \in H^2$ fixed:

$$\|S^{*n} C_\varphi S^n f\|^2 \leq \|\varphi^n (f \circ \varphi)\|^2$$

$$= \int_{\partial\mathbb{U}} |\varphi|^{2n} |f \circ \varphi|^2 \, dm$$

$$\to 0 \text{ by LDCT} \qquad ///$$

7. Converse of Thm. 1?

$\varphi(z) \not\equiv z$ from now on !!

Thm. 2 (Partial Converse)

$$\left.\begin{array}{c} C_\varphi \textit{ asymp. Toep.} \\ \\ \varphi(0) = 0 \end{array}\right\} \Rightarrow \; |\varphi| < 1 \textit{ a.e. on } \partial\mathbb{U}$$

Example. φ inner, $\varphi(0) = 0$

$$\Rightarrow$$

C_φ not asymp. Toeplitz on H^2.

Example. $\exists\varphi$ with $m\{|\varphi| = 1\} > 0$

but for which

C_φ *is* asymp. Toeplitz!

8. Proof of "Partial Converse"

Assume: $\varphi(0) = 0$ & $m\{|\varphi| = 1\} > 0$.

To Show: C_φ *not* Asymp. Toeplitz.

- C_φ asymp. Toep. $\Rightarrow S^{*n} C_\varphi S^n \to 0$ strongly on H^2 (later).

- $\varphi(0) = 0 \Rightarrow \psi(z) := \varphi(z)/z$, holo on $\mathbb{U}$.

$$\begin{aligned}
\|S^{*n}\, C_\varphi\, S^n\, 1\|^2 &= \int_{\partial\mathbb{U}} |S^{*n}\varphi^n|^2\, dm \\
&= \int_{\partial\mathbb{U}} |S^{*n}\, z^n\, \psi^n|^2\, dm \\
&= \int_{\partial\mathbb{U}} |\psi|^{2n}\, dm \\
&\geq m\{|\varphi| = 1\} > 0
\end{aligned}$$

///

Summary: If $\varphi(0) = 0$ then

C_φ asymp. Toep. $\Longleftrightarrow m\{|\varphi| = 1\} = 0$.

9. "Mean" Asymp. Toeplitzness

Theorem 3. $\forall\ \varphi, \quad \forall \alpha > 0$*:*

C_φ is "(C,α)–asymp. Toeplitz."

<u>*in fact*</u>: $\varphi(z) \not\equiv z \quad \Rightarrow \|S^{*n} C_\varphi S^n f\| \xrightarrow{(C,\alpha)} 0.$

- $s_n \xrightarrow{(C,1)} s$ means $\displaystyle \frac{1}{n+1} \sum_{j=0}^{n} s_j \to s$.

- $s_n \xrightarrow{(C,\alpha)} s$ means $\displaystyle \sum_{j=0}^{n} c_{nj}^{(\alpha)} s_j \to s$.

- $C_\alpha \neq C_1^\alpha$, *but* $C_\alpha \approx C_1^\alpha \qquad (\alpha = 2, 3, \ldots)$.

- $(C,\alpha) \searrow$ as $\alpha \searrow$

10. Matrix Convergence Methods

Defn. $A = [a_{ij}]_{i,j=0}^{\infty}$ is a *regular matrix*

means

$$\lim_n s_n = s \quad \Rightarrow \quad \lim_n \sum_j a_{n,j}\, s_j = s$$

Classical Thm. A *regular iff*

(i) $\lim_n a_{n,j} = 0 \qquad \forall\, j$ (col's $\to 0$)

(ii) $\sup_n \sum_j |a_{n,j}| < \infty$ ("absol. row sums" bndd)

(iii) $\lim_n \sum_j a_{n,j} = 1$ (row sums $\to 1$)

Defn. Call A V-regular if, in addition,

(iv) $\lim_n \sum_j |a_{n,j} - a_{n,j+1}| = 0$ (row varn's $\to 0$)

Rmk. C_α *is V-regular* $\forall\, \alpha > 0$.

11. A-(asymp) Toeplitzness

Defn. "T is *A-Toeplitz*" means

$$\left(\sum_j a_{n,j}\ S^{*n}\ T\ S^n\right)_{n=0}^{\infty}$$

strongly convergent on H^2.

Theorem 4. *If A is V-regular then every composition operator is A-Toeplitz on H^2.*

Proof outline.

- $S^{*n}\ C_\varphi\ S^n \quad = \quad T_{(\bar{z}\varphi)^n}\ C_\varphi \quad = \quad T_{\psi^n}\ C_\varphi$

- $\sum_j a_{n,j}\ S^{*n} T S^n \ = \ T_{\Psi_n} C_\varphi$

  where $\Psi_n = \sum_j a_{n,j}\ \psi^j$

Theorem 4. *If A is V-regular then every composition operator is A-Toeplitz on H^2.*

Proof outline.

- $S^{*n}\ C_\varphi\ S^n \quad = \quad T_{(\overline{z}\varphi)^n}\ C_\varphi \quad = \quad T_{\psi^n}\ C_\varphi$

- $\sum_j\ a_{n,j}\ S^{*n}TS^n\ =\ T_{\Psi_n}C_\varphi, \quad \Psi_n = \sum_j\ a_{n,j}\ \psi^j$

Therefore for $f \in H^2$:

- $\|\sum_j\ a_{n,j}\ S^{*n}TS^n f\| = \|T_{\Psi_n}C_\varphi f\| \leq \|\Psi_n C_\varphi f\|$

So *enough to show:*

- $\|\Psi_n f\| \to 0 \quad \forall\ f \in H^2$

- $\Psi_n\ =\ \dfrac{1}{(1-\psi)}\left(\sum_j (a_{n,j} - a_{n,j-1})\ \psi^n + a_{n,0}\right)$

Theorem 4. *If A is V-regular then every composition operator is A-Toeplitz on H^2.*

Enough to show:

- $\|\Psi_n f\| \to 0 \quad \forall\ f \in H^2$, where

  $$\Psi_n := \sum_j a_{n,j}\ \psi^j = \frac{1}{(1-\psi)}\left(\sum_j (a_{n,j} - a_{n,j-1})\ \psi^n + a_{n,0}\right),$$

  and $\psi(z) = \overline{z}\varphi(z) \qquad (z \in \partial\mathbb{U})$.

- $|\Psi_n| = |\sum_j a_{n,j}\ \psi^j| \leq \sum_j |a_{n,j}| \leq M$

- $|\Psi_n| \leq \dfrac{1}{|1-\psi|}\left(\sum_j (|a_{n,j} - a_{n,j-1}|)\ \psi^n + |a_{n,0}|\right)$

- $\Psi_n \to 0$ "boundedly a.e." on $\partial\mathbb{U}$.

- $\|\Psi_n f\|^2 = \int_{\partial\mathbb{U}} |\Psi_n|^2 |f|^2 \to 0 \qquad ///$

Have proved: *If A is V-regular, then:*

Theorem 4. *Every comp. operator is A-Toeplitz on H^2.*

Corollary. *Every comp. operator matrix has A-convergent diagonals.*

Qn. 1. Does every comp. operator matrix have *convergent* diagonals?

Equivalently: Is every comp. operator on H^2 "weakly asymptotically Toeplitz?"

Qn. 2. Which comp. ops. on H^2 are asymptotically Toeplitz?