Convergence to the Denjoy-Wolff point

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1. *The Denjoy-Wolff point*

$\varphi : \mathbb{U} \to \mathbb{U}$ holomorphic

- *Schwarz Lemma:*

$$\left.\begin{array}{l} \omega \in \mathbb{U} \\ \\ \varphi(\omega) = \omega \end{array}\right\} \Longrightarrow \quad \varphi_n \to \omega \text{ on } \mathbb{U}$$

- *Denjoy-Wolff Theorem (1920's)*

  If φ fixes <u>no</u> point of $\mathbb{U}$ then

  $\exists!\ \omega \in \partial\mathbb{U}$ such that

$$\varphi_n \to \omega \quad \text{on } \mathbb{U}$$

2. *Denjoy-Wolff: Complete Story*

$$\varphi : \mathbb{U} \to \mathbb{U} \qquad \begin{cases} \text{holomorphic} \\ \\ \text{no fixed point in } \mathbb{U} \end{cases}$$

($\star$) Julia's Lemma:

$\exists$ (!) $\omega \in \partial\mathbb{U}$ such that

$\forall H$: $\quad \varphi(H) \subset H$

- $\omega =$ DW point of φ.

- ω a "fixed point" of φ

  $\angle \lim_{z \to \omega} \varphi(z) = \omega$

- φ "conformal at ω"

  $$\angle \lim_{z \to \omega} \varphi'(z), \quad \angle \lim_{z \to \omega} \frac{\varphi(z) - \omega}{z - \omega}, \quad \angle \lim_{z \to \omega} \frac{1 - |\varphi(z)|}{1 - |z|}$$

  all exist, all equal.

3. *Denjoy-Wolff & Hardy Space*

H^2: All $f(z) = \sum_0^\infty a_n z^n \in \text{Hol}(\mathbb{U})$ such that

$$\|f\|^2 := \sum_0^\infty |a_n|^2 < \infty$$

A Hilbert space

"*Boundary H^2:* $f \in H^2 \Rightarrow$

- $\angle \lim_{z \to \zeta} f(z) = f(\zeta)$, a.e. $\zeta \in \partial\mathbb{U}$.

- $\|f\|^2 = \int_{\partial\mathbb{U}} |f(\zeta)|^2 \, dm(\zeta)$

DW Theorem (restated). $\forall \varphi : \mathbb{U} \to \mathbb{U}$ holo.

$\exists \omega \in \overline{\mathbb{U}}$ such that

$\varphi_n \to \omega$ *weakly* in H^2

(i.e. $\langle \varphi_n, f \rangle \to \langle \omega, f \rangle \quad \forall f \in H^2$)

4. *QN:* $\|\varphi_n - \omega\| \to 0$ *??*

NO if φ *inner* and $\omega \in \mathbb{U}$

Theorem. YES in every other case!

Proof. (a) $\omega \in \mathbb{U}$, φ not inner.

WLOG $\omega = 0$ (i.e., $\varphi(0) = 0$).

KNOWN: $\exists\ 0 < \delta < 1$ such that

$$\left.\begin{array}{l} f \in H^2 \\ \\ f(0) = 0 \end{array}\right\} \implies \|f \circ \varphi\| \le \delta \|f\|$$

$$\implies \|\varphi_n\| \le \delta^n \to 0 \qquad \square$$

We are proving

<u>Theorem</u>. $\|\varphi_n - \omega\| \to 0$

UNLESS φ inner & $\omega \in \mathbb{U}$.

Proof. (a) $\omega \in \mathbb{U}$, φ not inner: Done!

(b) $\omega \in \partial\mathbb{U}$, WLOG $\omega = 1$.

$$\begin{aligned}
\|\varphi_n - 1\|^2 &= \|f \circ \varphi_n\|^2 \qquad (f(z) = z - 1) \\
&= \int |f \circ \varphi_n|^2 \, dm \\
&= P[|f \circ \varphi_n|^2](0) \\
&\leq P[|f|^2](\varphi_n(0)) \\
&= P[2(1 - \operatorname{Re} z)](\varphi_n(0)) \\
&= 2(1 - \operatorname{Re} \varphi_n(0)) \\
&\to 0 \qquad \square
\end{aligned}$$

5. QUESTION

$$\sim \left\{ \begin{array}{c} \varphi \text{ inner} \\ \& \\ \omega \in \mathbb{U} \end{array} \right\} \quad \overset{???}{\Longrightarrow} \quad \varphi_n \longrightarrow \omega \text{ a.e. on } \partial\mathbb{U}$$

Remarks.

- $(\star) \Rightarrow \varphi_{n_k} \to \omega$ a.e. on $\partial\mathbb{U}$.
- $\varphi_n \to \omega$ a.e. $\Rightarrow \|\varphi_n - \omega\| \to 0$

Proposition

$$\left\{ \begin{array}{c} \varphi \text{ not inner} \\ \& \\ \omega \in \mathbb{U} \end{array} \right\} \quad \Longrightarrow \quad \varphi_n \longrightarrow \omega \text{ a.e. on } \partial\mathbb{U}$$

Proof. WLOG $\omega = 0$.

$\|\varphi_n\| < \delta^n$ some $0 < \delta < 1$

$$\infty > \sum \|\varphi_n\|^2 = \sum \int |\varphi_n|^2 \, dm = \int \left(\sum |\varphi_n|^2\right) dm$$

$\Rightarrow \sum |\varphi_n|^2 < \infty$ a.e. $\Rightarrow \varphi_n \to 0$ a.e. $\square$

Recall our question:

$$\sim \left\{ \begin{array}{c} \varphi \text{ inner} \\ \& \\ \omega \in \mathbb{U} \end{array} \right\} \quad \overset{???}{\Longrightarrow} \quad \varphi_n \longrightarrow \omega \text{ a.e. on } \partial\mathbb{U}$$

So far have proved:

$$\left\{ \begin{array}{c} \varphi \text{ not inner} \\ \& \\ \omega \in \mathbb{U} \end{array} \right\} \quad \Longrightarrow \quad \varphi_n \longrightarrow \omega \text{ a.e. on } \partial\mathbb{U}$$

6. QUESTION: What if $\omega \in \partial\mathbb{U}$?

Know $\varphi_n \to \omega$ in norm. ?? $\to$ A.E ??

$$\|\varphi_n - 1\|^2 \leq 2(1 - \operatorname{Re} \varphi_n(0))$$

$$\leq 2(1 - |\varphi_n(0)|^2)$$

$$\boxed{\sum (1 - |\varphi_n(0)|) < \infty \Rightarrow \varphi_n \to \omega \text{ a.e.}}$$

Our Question: "$\varphi_n \to \omega$ a.e." ??

So far:

- FALSE if φ inner and $\omega \in \mathbb{U}$.

- TRUE if either:

  (i) $\omega \in \mathbb{U}$ and φ not inner

  (ii) $\sum(1 - |\varphi_n(0)|) < \infty$

Remarks: For $\omega \in \partial\mathbb{U}$:

- $\sim [\varphi_n \to \omega$ a.e. $\Rightarrow$ (ii)]

- (ii) $\Rightarrow \varphi_n \to \omega$ a.e. for φ inner!

  (Doering & Mañe, 1991)

Question: $\sum(1 - |\varphi_n(0)|) < \infty$

For which φ with $\omega \in \mathbb{U}$?

7. Question: $\sum(1-|\varphi_n(0)|)<\infty$

for which φ with $\omega \in \partial\mathbb{U}$?

Two "types" of φ with $\omega \in \partial\mathbb{U}$:

- *Hyperbolic:* $\varphi'(\omega) < 1$

- *Parabolic:* $\varphi'(\omega) = 1$

Proposition

φ *hyperbolic* $\Rightarrow \sum(1-|\varphi_n(0)|)<\infty$

$$(\Rightarrow \varphi_n \to \omega \text{ a.e.})$$

Proof. Julia's Lemma again!

In Right Half-Plane:

$\Phi(\{\text{Re}\, w > \delta\})$

$\subset \{\text{Re}\, w > \delta/\varphi'(\omega)\}$

Proposition

φ *hyperbolic* $\Rightarrow \sum(1 - |\varphi_n(0)|) < \infty$

$$(\Rightarrow \varphi_n \to \omega \text{ a.e.})$$

Proof. Julia's Lemma again!

In $\mathbb{U}$:

$$|\omega - \varphi_n(0)|^2 \leq \varphi'(\omega)^n(1 - |\varphi_n(0)|^2) \leq \varphi'(\omega)^n.$$

Thus

$$1 - |\varphi_n(0)| \leq |\omega - \varphi_n(0)| \leq \varphi'(\omega)^{n/2}$$

hence (because $\varphi'(\omega) < 1$)

$$\sum_n (1 - |\varphi_n(0)|) < \infty \qquad \square$$

Fund'l Question: $\varphi_n \to \omega$ a.e. for which φ?

So far:

- False if φ inner, $\omega \in \mathbb{U}$.

- True if:
  φ not inner, $\omega \in \mathbb{U}$, or

  φ of hyperbolic type
  $(\omega \in \partial\mathbb{U},\ \varphi'(\omega) < 1)$.

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8. Question: $\varphi_n \to \omega$ for φ of *parabolic type*? $(\varphi'(\omega) = 1)$

Parabolic Dichotomy:

ψ-hyp. metric: $\rho(z, w) := \left|\dfrac{z - w}{1 - \overline{z}w}\right|$

Say φ is of:

- *Automorphism type* if
  $$\lim_n \rho(\varphi_n(0), \varphi_{n+1}(0)) > 0$$

- *Non-automorphism type* if
  $$\lim_n \rho(\varphi_n(0), \varphi_{n+1}(0)) = 0$$

Proposition φ of parabolic auto. type

$$\Rightarrow \quad \sum_n (1 - |\varphi_n(0)| < \infty$$

($\Rightarrow \varphi_n \to \omega \in \partial\mathbb{U}$ a.e.)

Proof. Linear Fractional Model Theorem
(Baker, Cowen, Pommerenke, 1980's)

$\exists\ \sigma : \mathbb{U} \to$ RHP holomorphic such that

$$\sigma \circ \varphi = \sigma + ib \qquad (\exists\ b \in \mathbb{R}).$$

$$\sigma \circ \varphi_n = \sigma + inb \qquad (n = 1, 2, \ldots)$$

$(\sigma(0) + inb)$ a Blaschke sequence for RHP
$\Rightarrow$
$(\sigma(0)+inb) =$ zero-set for some $F \in H^\infty$(RHP)
$\Rightarrow$
$(\varphi_n(0)) =$ zero-set of $f := F \circ \sigma \in H^\infty(\mathbb{U})$
$\Rightarrow$
$\sum_n (1 - |\varphi_n(0)| < \infty$ $\square$

9. Fund'l Question: $\varphi_n \to \omega$ a.e.??

So far:

For $\omega \in \mathbb{U} \backslash \{$"usual suspects"$\}$: YES

For $\omega \in \partial\mathbb{U}$—three classes of maps:

- Hyperbolic type ($\varphi'(\omega) < 1$): YES
- Parabolic auto type: YES
  ($\varphi'(\omega) = 1$, orbits separated)

- Parabolic non-auto type: YES & NO
  ($\varphi'(\omega) = 1$, orbits not separated)

YES: Parabolic, non-automorphic, LFT $\mathbb{U} \to \mathbb{U}$

NO: Singular inner function

$$\varphi(z) = \exp\left\{2\,\frac{z-1}{z+1}\right\}, \qquad \omega = 1$$

$$\sum_n (1 - |\varphi_n(0)| = \infty \Rightarrow\ \sim [\varphi_n \to \omega \quad \text{a.e.}\ ]$$

10. Inner Functions & Ergodic Theory

φ *inner:* $\partial\mathbb{U} \to \partial\mathbb{U}$

Definition. φ *ergodic* means:

$f \circ \varphi = f \quad \Rightarrow \quad f \equiv$ const. a.e

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Theorem (Aaronson, Neuwirth, Pommerenke):

$$\varphi \text{ ergodic on } \partial\mathbb{U} \iff \begin{cases} \omega \in \mathbb{U}, \quad \text{or} \\ \\ \varphi \text{ of non-auto type} \\ \quad (\omega \in \partial\mathbb{U}, \varphi'(\omega) = 1, \\ \quad \text{orbits non-sep}) \end{cases}$$

Invariant measure ($d\mu\varphi^{-1} = d\mu$):

- $m_\omega := P_\omega \, dm$ if $\omega \in \mathbb{U}$ (Nordgren '68)
- $\varphi'(\omega) \dfrac{dm(\zeta)}{|\omega - \zeta|^2}$ if $\omega \in \partial\mathbb{U}$ (Letac '77)

Birkhoff Ergodic Th. For finite mpt T:

T ergodic $\Rightarrow f \circ T^n \to \int f \, d\mu$ (C,1) $\forall f \in L^1(\mu)$

Birkhoff Ergodic Th. For finite mpt T:

$$T \text{ ergodic } \Rightarrow f \circ T^n \to \int f\, d\mu \text{ (C,1) } \forall f \in L^1(\mu)$$

Apply Birkhoff w/ $f(\zeta) \equiv \zeta$
to φ with $\omega \in \mathbb{U}$:

$$\varphi_n \to \int \zeta\, dm_\omega(\zeta) = \omega \quad \text{(C,1)}$$

Conclude (φ_n) does not converge a.e. on $\partial\mathbb{U}$.

Question. Also true for ergodic φ with $\omega \in \partial\mathbb{U}$?

Example. $Tx = x + 1$ on $\mathbb{Z} \cup \{\infty\}$
$\mu =$ counting measure on $\mathbb{Z}$, $\mu\{\infty\} = 0$

T ergodic, but $T^n \to \infty$ a.e.

Aaronson (1978) NO:
$\exists\ \varphi$ ergodic (orbits nonsep.)
with $\varphi_n \to \omega$ a.e. $(\sum_n (1 - |\varphi_n(0)| < \infty)$