Essentially normal composition operators

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Operator theoretic setting

- $\mathcal{H}$: A Hilbert space
- $\mathcal{L}(\mathcal{H})$: All bounded (i.e., continuous) linear operators on $\mathcal{H}$.
- *Adjoint* $T \in \mathcal{L}(\mathcal{H})$: $T^* \in \mathcal{L}(\mathcal{H})$ defined by:

$$\langle T^* f, g\rangle := \langle f, Tg\rangle \qquad (f, g \in \mathcal{H})$$

- $T \in \mathcal{L}(\mathcal{H})$ is
  - *normal* if: $T^*T = TT^*$
  - *essentially normal* if:

$$[T^*, T] := T^*T - TT^*$$

  is *compact*.

Examples of ess. normal operators

- *normal, compact:*
  (“trivially” essentially normal)

- *normal plus compact*

- *Shifts on ℓ^2:* For $x = (x(1), x(2), \ldots) \in \ell^2$

  - *Forward shift* $Sx = (0, x(1), x(2), \ldots)$

  - *Backward shift* $Bx = (x(2), x(3), \ldots)$

  - Prop. S *and* B *are essentially normal*

    $S^* = B$

    $S^*S = I$

    $SS^*x = (0, x(2), x(3), \ldots)$

    $[S^*, S]x := (S^*S - SS^*)x = (x(0), 0, 0, \ldots)$
    rank one, so *compact*

  Rmk. S, B *not* “normal plus compact”

Function theoretic setting

- $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$

- *The Hardy space H^2:* All functions

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n)\ z^n \ \in \ \mathrm{Hol}\,(\mathbb{U})$$

  such that: $\|f\|^2 := \sum |\hat{f}(n)|^2 < \infty$.

- *Inner product:*

$$\langle f, g \rangle := \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$$

- $\varphi : \mathbb{U} \to \mathbb{U}$ holomorphic "selfmap" of $\mathbb{U}$.

- $C_\varphi f = f \circ \varphi \qquad (f \ \in \ \mathrm{Hol}\,(\mathbb{U}))$

- $C_\varphi \in \mathcal{L}(H^2)$ (Littlewood 1925).

Which C_φ are essentially normal?

- *Example:* Complex dilations.

  Fix $a \in \overline{\mathbb{U}}$

  Define $\varphi_a(z) := az \qquad (z \in \mathbb{U})$

  $\varphi_a(z^n) = a^n z^n$

  $[C_{\varphi_a}] = \text{diag}\,\{1, a, a^2, \ldots\}$,

  so C_{φ_a} *normal*.

- *H. J. Schwartz, 1969:*
  C_φ normal $\Rightarrow \varphi = \varphi_a, \quad \exists a \in \overline{\mathbb{U}}$.

Proof of Schwartz's Theorem

To Show: C_φ *normal* $\Rightarrow \varphi = \varphi_a, \quad \exists a \in \overline{\mathbb{U}}$.

(a) General fact about normal operators:

T normal on $\mathcal{H}$

$\Rightarrow \ker T = \ker T^*$

$\Rightarrow$ If $Tf = \lambda f$ then $T^* f = \overline{\lambda} f$

Proof.
Will show: $\|Tf\| = \|T^*f\| \quad \forall f \in \mathcal{H}$!!

$$\begin{aligned} \|Tf\|^2 &= \langle Tf, Tf\rangle = \langle T^*Tf, f\rangle \\ &= \langle TT^*f, f\rangle \qquad (T \text{ normal}) \\ &= \langle T^*f, T^*f\rangle = \|T^*f\|^2 \end{aligned}$$

□

(b) C_φ *normal* $\Rightarrow \varphi(0) = 0$.

Proof. $C_\varphi 1 = 1$ so $C_\varphi^* 1 = 1$ (by (a)).

$0 = \langle z, 1\rangle = \langle z, C_\varphi^* 1\rangle = \langle C_\varphi z, 1\rangle = \langle \varphi, 1\rangle = \varphi(0)$.

$\square$

(c) $\varphi(z) = \varphi'(0)z \qquad (\forall z \in U)$

Proof. $\forall z \in H^2$:

$$\begin{aligned} \langle f, C_\varphi^* z\rangle &= \langle C_\varphi f, z\rangle = \widehat{f \circ \varphi}(1) \\ &= (f \circ \varphi)'(0) = f'(\underbrace{\varphi(0)}_{=0})\varphi'(0) \\ &= \langle \varphi'(0) f, z\rangle = \langle f, \overline{\varphi'(0)} z\rangle \end{aligned}$$

$\Rightarrow C_\varphi^* z = \overline{\varphi'(0)} z$

$\Rightarrow \underbrace{C_\varphi z}_{\varphi(z)} = \varphi'(0) z \qquad$ (by (a) & normality).

$\square$

Which C_φ are (nontrivially) essentially normal?

<u>Notation</u>

- HSM$\,(\mathbb{U})$: All holom. selfmaps $\varphi : \mathbb{U} \to \mathbb{U}$.

- LFT$\,(\mathbb{U})$: All *linear-fractional* $\varphi \in$ HSM$\,(\mathbb{U})$
$$\varphi(z) = \tfrac{az+b}{cz+d} \qquad (ad - bc \neq 0)$$

- Aut$\,(\mathbb{U})$: All $\varphi \in$ LFT$\,(\mathbb{U})$ with $\varphi(\mathbb{U}) = \mathbb{U}$.

<u>Nina Zorboska (1999)</u>

- $\varphi \in$ Aut$\,(\mathbb{U})$, *ess. normal*
$$\Rightarrow \varphi = \varphi_a, \quad \exists\, |a| = 1$$
$$(\varphi_a(z) \equiv az)$$

- <u>Question.</u> $\exists$? *nontriv. ess. normal* C_φ?

- *For $\varphi \in$ HSM$\,(\mathbb{U})$:*
C_φ nontriv. ess. normal $\Rightarrow$
φ has no fixed point in $\mathbb{U}$.

Main Theorem (BLNS 2003).

For $\varphi \in \text{LFT}(\mathbb{U})$*:*

$$C_\varphi \text{ nontrivially essentially normal} \iff \varphi \text{ a parabolic non-automorphism.}$$

Parabolic non-automorphisms

$\varphi_t(z) = \frac{(2-t)z+t}{-tz+(2+t)}$ $\qquad$ $\varphi_2(z) = \frac{1}{2-z}$

Cowen's Adjoint Formula (1988)

For $\varphi(z) = \frac{az+b}{cz+d} \in \mathrm{LFT}(\mathbb{U})$*:*

$$C_\varphi^* = M_g\, C_\sigma\, M_h^*$$

where

$$\sigma := \rho \circ \varphi^{-1} \circ \rho \in \mathrm{LFT}(\mathbb{U}),$$

$$g(z) := \frac{1}{-\bar{b}z + \bar{d}} \in H^\infty,$$

$$h(z) := cz + d \in H^\infty.$$

Commutator Formula

For $\varphi \in \mathrm{LFT}(\mathbb{U})$*:*

$$[C_\varphi^*, C_\varphi] := C_\varphi^* C_\varphi - C_\varphi C_\varphi^*$$

$$= M_g\,[C_\sigma, C_\varphi]\,M_h^* + M_g\,C_\sigma\,[M_h^*, C_\varphi]$$

$$+ (M_g - M_{g\circ\varphi})\,C_{\sigma\circ\varphi}\,M_h^*$$

<u>Corollary.</u> *For* $\varphi \in \mathrm{LFT}(\mathbb{U}) \backslash \mathrm{Aut}(\mathbb{U})$

C_φ *essentially normal* $\iff$ $[C_\sigma, C_\varphi]$ *compact*

(where $\sigma := \rho \circ \varphi^{-1} \circ \rho$)

<u>Corollary.</u>

φ *parabolic non-automorphism*

$\Rightarrow$

C_φ *essentially normal.*

<u>To show:</u> $[C_\sigma, C_\varphi]$ compact.

φ has one fixed pt. in $\widehat{\mathbb{C}}$ (on $\partial\mathbb{U}$)

Same for $\sigma := \rho \circ \varphi^{-1} \circ \rho$

σ parabolic <u>with same FP as φ</u>.

$\sigma \circ \varphi = \varphi \circ \sigma$

$[C_\sigma, C_\varphi] = 0$ $\square$

Remains to show:

$\varphi \in \text{LFT}(\mathbb{U}) \backslash \text{Aut}(\mathbb{U})$

No FP in $\mathbb{U}$	$\Rightarrow$	C_φ *not essentially normal*
Not parabolic		

The key:

$\varphi \circ \sigma$ and $\sigma \circ \varphi$ parabolic
with same fixed point.

$\varphi \circ \sigma = \varphi_t$ and $\sigma \circ \varphi = \varphi_s$, $\exists s, t \in$ RHP.

Eigenvalues

$$f_\lambda(z) = \exp\left\{ -\lambda \underbrace{\frac{1+z}{1-z}}_{\sigma(z)} \right\} \in H^\infty \qquad \forall \lambda > 0$$

$$C_{\varphi_t}(f_\lambda) = e^{\lambda t} f_\lambda$$

Eigenvalues

$$f_\lambda = \exp\left\{-\lambda\tfrac{1+z}{1-z}\right\} \in H^\infty \qquad \forall\lambda > 0$$

$$C_{\varphi_t}(f_\lambda) = e^{\lambda t} f_\lambda$$

$$[C_\varphi, C_\sigma]f_\lambda = (C_{\varphi_t} - C_{\varphi_s})f_\lambda = (e^{-\lambda t} - e^{-\lambda s})f_\lambda$$

$$\text{spectrum}\,([C_\varphi, C_\sigma]) \supset \underbrace{\{e^{-\lambda t} - e^{-\lambda s} : \lambda > 0\}}_{\text{curve in } \mathbb{C}\ (s \neq t)}$$

Conclude:

- $[C_\varphi, C_\sigma]$ not compact (Riesz Theory of compact operators)!
- C_φ not essentially normal. □