Hardy spaces that support no compact composition operators

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$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$

The Hardy space H^2: All functions

$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathrm{Hol}\,(\mathbb{U})$$

such that:

$$\|f\|^2 := \sum |\widehat{f}(n)|^2 < \infty.$$

Equivalently:

$$\|f\|^2 = \sup_{0 \le r < 1} \int_{|z|=r} |f(z)|^2 \, |dz| < \infty.$$

Composition Operators

$\varphi : \mathbb{U} \to \mathbb{U}$ holomorphic

C_φ : Hol $(\mathbb{U}) \to$ Hol $(\mathbb{U})$

Littlewood (1925):

$C_\varphi : H^2 \to H^2$

The Compactness Problem:

"Which C_φ are compact on H^2 ?"

H.J. Schwartz 1968: Initial results.

JHS 1987: Char'n via value distribution of φ

Examples

(A) Compact:

$\varphi \equiv a \in \mathbb{U}$

$\|\varphi\|_\infty < 1$

(e.g. $\varphi(z) = z/2$).

$\varphi(\mathbb{U}) \subset$ inscribed polygon

(B) Not compact:

$$\varphi(z) = \frac{1+z}{2}$$

Question for today:

"For which simply connected $G \subset \mathbb{C}$ does $H^2(G)$ support compact composition operators?"

Two kinds of $H^2(G)$:

(I) Conformally Invariant $H^2(G)$

$$F \in H^2(G) \Longleftrightarrow F \circ \tau \in H^2(\mathbb{U})$$

$C_\tau : H^2(G) \to H^2(\mathbb{U})$ isometry (onto)

(I) Conformally Invariant $H^2(G)$

$$
\begin{array}{ccc}
H^2(G) & \xrightarrow{\quad C_\Phi \quad} & H^2(G) \\
C_\tau \Big\uparrow & & \Big\uparrow C_\tau \\
H^2(\mathbb{U}) & \xrightarrow[C_\varphi \,=\, C_\tau^{-1} C_\Phi C_\tau]{} & H^2(\mathbb{U})
\end{array}
$$

(II) <u>Non-Conformally-Invariant $H^2(G)$</u>

$H^2(G)$ = all $F \in \mathrm{Hol}(\mathbb{U})$ with

$$\|F\|^2 := \sup_{0 \le r < 1} \int_{\tau(\{|z|=r\})} |F(w)|^2 \, |dw| < \infty$$

$$= \sup_{0 \le r < 1} \int_{(\{|z|=r\}} |F(\tau(z))|^2 \, |\tau'(z)| \, |dz|$$

i.e., $F \in H^2(G) \Longleftrightarrow (F \circ \tau)(\tau')^{1/2} \in H^2(\mathbb{U})$

& linear map

$$V : H^2(G) \to H^2(\mathbb{U})$$

an <u>onto isometry</u>.

Fundamental Example

$G =$ RHP, the right half-plane

$$F \in H^2(\text{RHP}) \iff \frac{1}{1-z} F\left(\frac{1+z}{1-z}\right) \in H^2$$

$$\iff \sup_{x>0} \int_{-\infty}^{\infty} |F(x+iy)|^2 \, dy < \infty$$

Valentin Matache (PAMS 1999):

No *compact comp. op on* H^2(RHP)*!!!*

Main Theorem
(S & S 2002):

$H^2(G)$ supports

compact comp. ops

$$\Longleftrightarrow$$

∂G has finite length

Retreat to $H^2(\mathbb{U})$

$$
\begin{array}{ccc}
G & \xrightarrow{\Phi} & G \\
\tau \big| & & \big| \tau \\
\mathbb{U} & \xrightarrow[\varphi]{} & \mathbb{U}
\end{array}
\qquad\qquad
\begin{array}{ccc}
H^2(G) & \xrightarrow{C_\Phi} & H^2(G) \\
V \big| & & \big| V \\
H^2(\mathbb{U}) & \xrightarrow[A_\varphi]{} & H^2(\mathbb{U})
\end{array}
$$

$$VF = (\tau')^{1/2}(F \circ \tau)$$

$$\Rightarrow$$

$$A_\varphi f = \left(\frac{\tau'}{\tau' \circ \varphi}\right)^{1/2}(f \circ \varphi)$$

Weighted comp. op: $H^2(\mathbb{U}) \to H^2(\mathbb{U})$

unitarily equiv. to $C_\Phi : H^2(G) \to H^2(G)$.

Tools for the proof

(I) Privalov-Smirnov-Pommerenke:

$$\partial G \text{ has finite length} \iff \tau' \in H^1(\mathbb{U})$$

(II) Thm (Caughran & Schwartz for $\mathbb{U}$, 1975):

C_Φ *compact on* $H^2(G) \Rightarrow \Phi$ *fixes a pt. of* G.

A_φ *compact on* $H^2(\mathbb{U}) \Rightarrow \varphi$ *fixes a pt. of* $\mathbb{U}$.

(III) Riesz Theory of Compact Operators:

T *compact* $\Rightarrow \sigma(T)\backslash\{0\}$ *eigenvalues.*

Main Thm

$$H^2(G) \text{ *has compact* } C_\Phi\text{'s}$$

$$\Longleftrightarrow$$

$$\tau' \in H^1(\mathbb{U})$$

(A) *Easy part:*

$$\tau' \in H^1(\mathbb{U}) \Rightarrow \exists \;\text{ compact } C_\Phi \text{ on } H^2(G)$$

Proof. Claim any $\Phi \equiv$ const. $\in G$ works.

$\varphi \equiv a \in \mathbb{U} \Rightarrow$

$$A_\varphi f = \left(\tfrac{\tau'}{\tau' \circ \varphi}\right)^{1/2} f \circ \varphi = \underbrace{(\tau')^{1/2}}_{\in H^2} \underbrace{\left(\frac{f(a)}{\tau'(a)^{1/2}}\right)}_{\text{bndd lin fnl}}$$

i.e. A_φ a bounded rank 1 op. on $H^2(\mathbb{U})$. $\square$

(B) *Main Theorem: Hard part:*

Given: $C_\Phi : H^2(G) \to H^2(G)$ compact

i.e. $A_\varphi : H^2(\mathbb{U}) \to H^2(\mathbb{U})$ compact.

To show: $\tau' \in H^1$

First will show:

$$\textit{1 an eigenvalue of } A_\varphi$$

$$\exists f \in H^2(\mathbb{U})\backslash\{0\} \text{ with } A_\varphi f = f$$

Then will show:

$$f = \text{const. } (\tau')^{1/2}$$

So $\tau' \in H^1$ and done (by Privalov et al).

WLOG: $\varphi(0) = 0$

(I) $A_\varphi^* 1 = 1$

Proof. $\forall f \in H^2(\mathbb{U})$:

$$\begin{aligned}
\langle f, A_\varphi^* 1\rangle &= \langle A_\varphi f, 1\rangle = \left\langle \left(\tfrac{\tau'}{\tau'\circ\varphi}\right)^{1/2} (f\circ\varphi), 1\right\rangle \\
&= \left(\frac{\tau'(0)}{\tau'(\varphi(0))}\right)^{1/2} f(\underbrace{\varphi(0)}_{=0}) = f(0) \\
&= \langle f, 1\rangle
\end{aligned}$$

□

(II) *1 an eigenvalue of* A_φ

Proof.

(I) $\Rightarrow$ $1 \in \sigma(A_\varphi^*)$

$\Rightarrow$ $1 \in \sigma(A_\varphi)$

$\Rightarrow$ 1 an eigenvalue of A_φ (Riesz Th.) □

So far: (I) & (II) $\Rightarrow$:

$$\exists f \in H^2(\mathbb{U})\backslash\{0\} \text{ with } A_\varphi f = f$$

(III) To show: $f = \text{const. } (\tau')^{1/2}$

So $\tau' \in H^1$ and *done* (by Privilov et al).

Proof. $f = A_\varphi f = A_\varphi^n f = A_{\varphi_n} f$

Now fix $z \in \mathbb{U}$:

$\varphi_n(z) \to 0$ as $n \to \infty$, hence

$$f(z) \;=\; \left(\frac{\tau'(z)}{\tau'(\varphi_n(z))}\right)^{1/2} f(\underbrace{\varphi_n(z)}_{\to 0})$$

$$\to\; (\tau'(z))^{1/2} \frac{f(0)}{\underbrace{\tau'(0)^{1/2}}_{\neq 0}}$$

□

Bergman Spaces $A^2(G)$

$$F \in A^2(G) \iff \int_G |F(w)|^2 \, dA(w) < \infty$$

Thm. *For G simply connected:*

$A^2(G)$ *supports compact C_Φ's*

$$\iff$$

Area$(G) < \infty$

Question. *What if G not simply connected?*