Dynamical properties of linear operators

Cyclicity

- X: Banach space (separable)
- $T : X \to X$: bounded (i.e., continuous)
  linear operator on X.
- *T-orbit of vector $x \in X$* :

$$\mathrm{orb}_T(x) := \{x, Tx, T^2x, \cdots\}$$

- T is *cyclic* on X if $\exists x \in X$ with

$$\mathrm{span}\ \{\mathrm{orb}_T(x)\}$$

dense in X.

- Hypercyclic $\equiv$ Top. Transitive

Why study cyclicity?

- *Invariant Subspace Problem*

  "Does every $T : X \to X$ have
  nontrivial closed invariant subspace?"

  - $\overline{\text{span orb}_T(x)}$ = smallest T-invariant (closed) subspace that contains x.

  - T has nontriv. closed invar. subspace
    $\iff$
    T has a non-cyclic vector ($\neq 0$).

  - Enflo ($\approx$ 1980) *No* for some X, T.
  - Read ($\approx$ 1985) *No* for $X = c_0$, $X = \ell^1$.

  - For X = Hilbert space: still open!

- *Invariant <u>Subset</u> Problem*
  - "subspace" $\to$ "subset"
  - "cyclic $\to$ "hypercyclic"

Examples of HC operators

- *G.D. Birkhoff* (1929):

  $X = H(\mathbb{C})$ (all entire functions)

  $T : X \to X$ *"Translation by one"*

  $$Tf(z) = f(z+1)$$

  is hypercyclic on $H(\mathbb{C})$.

- *Rolewicz* (1969)
  - $X = \ell^2$
  - $B =$ Backward Shift on ℓ^2:

    $$B(\zeta_0, \zeta_1, \zeta_2, \cdots) = (\zeta_1, \zeta_2, \zeta_3, \cdots)$$

    (B *not* hypercyc: $\|Bx\| \leq \|x\|$)

  - THM. λB *is hypercyclic* $\forall\, |\lambda| > 1$

- *Godefroy-Shapiro* (1991):

  - Generalized Birkhoff

    $T : H(\mathbb{C}) \to H(\mathbb{C})$ *continuous, linear*

    $T \neq$ *const.* I

    TD = DT

    $\Rightarrow$

    T *hypercyclic*

  - Generalized Rolewicz to $f(B) \quad (f \in H^\infty)$

    $f(B)$ *HC on* $\ell^2 \iff f(\mathbb{U}) \cap \partial\mathbb{U} \neq \emptyset$

    (e.g., $I + B$ is HC, $2(I + B)$ is not)

- *Bourdon-Shapiro* (1997). Many composition operators on H^2 are hypercyclic!

- Finite dimensional case:
  NO HC OPERATORS

A Hypercyclicity criterion

(Kitai 1983, Gethner & JHS 1987)

Sufficient for $T : X \to X$ hypercyclic:

- $\exists\ Y, Z \subset X$ *dense sets*
- $\exists\ S : Z \to Z$ *a map*

such that:

- $T^n y \to 0 \quad \forall y \in Y$
- $S^n z \to 0 \quad \forall z \in Z$
- $TS = I_Z$

Proof of Rolewicz's Theorem

λB *is hypercylic* $\quad \forall\ |\lambda| > 1.$

$$Y \quad = \quad \{(\zeta_0, \zeta_1, \ldots \zeta_n, 0, 0, \ldots)\ \exists\ n\}.$$

$$Z \quad = \quad \ell^2$$

$$S \quad = \quad \frac{1}{\lambda}\ \text{(Forward shift)} \qquad \square$$

Proof of "a hypercyclicity criterion"

- Birkhoff's Transitivity Theorem:

  T *hypercyclic iff*

  $\forall\ U, V$ *open,* $\neq \emptyset$

  $\exists\ n$ *such that* $T^n(U) \cap V \neq \emptyset$

- Proof of HC criterion:

  1. Given U, V fix $y \in U \cap Y$ and $z \in V \cap Z$
  2. $T^n y \to 0$
     $S^n z \to 0$
  3. $x_n := y + S^n z \to y$
  4. $T^n x_n = T^n y + T^n S^n z$
     $= T^n y + z \to z \in V$
  5. $T^n x_n \in T^n(U) \cap V \neq \emptyset, \quad \exists n$ $\square$

- Baire Category Th. $\Rightarrow$ Birkhoff's Trans. Th.
  $\Rightarrow$ dense G_δ set of HC vectors!!

Remarkable Properties of HC operators

- (Ansari 1995)

  $$T \text{ HC} \Rightarrow T^n \text{ HC} \qquad \forall\, n = 2, 3, \ldots$$

- (Costakis–Peris $\approx$ 2000)

  "finitely HC" $\Rightarrow$ HC

  (both false for: "cyclic", "top. transitive.")

- (Bourdon–Feldman 2002)

  "locally HC" $\Rightarrow$ HC.

(some orbit somewhere dense)

Composition operators

- $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$

- The Hardy space H^2: *All fcns*

$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathit{Hol}\,(\mathbb{U})$$

  such that:

$$\|f\|^2 := \sum |\widehat{f}(n)|^2 < \infty.$$

- $C_\varphi : \mathrm{Hol}\,(\mathbb{U}) \to \mathrm{Hol}\,(\mathbb{U})$

- Littlewood (1925):

$$C_\varphi : H^2 \to H^2$$

  bounded linear operator

- Connection with dynamics:

$$C_\varphi^n = C_{\varphi_n}$$

Which C_φ are hypercyclic on H^2 ?

- Denjoy-Woff (1920's)
  Dynamics of $\varphi : \mathbb{U} \to \mathbb{U}$ *"trivial."*

  P. Bourdon-JHS (1990's +)
  "Dynamics of C_φ*'s on* H^2 <u>*not*</u> *trivial."*

- C_φ *hypercyclic on* H^2
  $\Rightarrow$
  (a) φ *has no fixed point in* $\mathbb{U}$
  (b) φ *univalent on* $\mathbb{U}$.

  Proof of (a).

  $\varphi(a) = a$ for some $a \in \mathbb{U}$
  $\Rightarrow$
  $\varphi_n(z) \to a \quad \forall\ z \in \mathbb{U}$
  $\Rightarrow$
  $C_\varphi^n f(z) = f(\varphi_n(z)) \to f(a) \quad \forall\ z \in \mathbb{U}$
  $\Rightarrow$
  $g(z) \equiv f(a) \quad \forall g \in \text{orb}_{\,C_\varphi}(f)$. $\square$

Linear-fractional C_φ's

- LFT ($\mathbb{U}$) := all linear-fractional $\varphi : \mathbb{U} \to \mathbb{U}$

- $\varphi \in$ LFT ($\mathbb{U}$) with no fixed point in $\mathbb{U}$:

  φ hyperbolic or parabolic
  (with fixed point on $\partial\mathbb{U}$).

- Theorem (Bourdon-JHS 1990's).

  For $\varphi \in$ LFT ($\mathbb{U}$), no FP in $\mathbb{U}$:

  C_φ *hypercyclic*

  $\iff$

  (a) φ hyperbolic, or
  (b) φ a parabolic automorphism

  (i.e. only parabolic non-auto's fail to be HC.)

- *φ parabolic auto.* $\Rightarrow C_\varphi$ *HC*

- Proof: Use "Hypercyclicity Criterion."

  Want to find

  – X dense in H^2 with $C_\varphi^n \to 0$ on X

  – Y dense in H^2 and $S : Y \to Y$ with $S^n \to 0$ on Y.

  – $C_\varphi S = I$ on Y.

- Take $S = C_\varphi^{-1} = C_{\varphi^{-1}}$

- Take $X = Y$
  = all polynomials p with $p(1) = 0$.

$$\|C_\varphi^n p\|^2 = \int_{\partial\mathbb{U}} |\underbrace{p(\underbrace{\varphi_n(z)}_{\to 1})}_{\to 0}|^2 \, dm(z) \to 0$$

(Lebesgue bounded convergence theorem.) □

φ hyperbolic $\Rightarrow C_\varphi$ HC:

- φ an auto: "same" as parabolic auto case
  - Two FP's on $\partial\mathbb{U}$:
    α (attractive), β (repulsive)
  - $S = C_\varphi^{-1} = C_{\varphi^{-1}}$
  - $X =$ all polys p with $p(\alpha) = 0$
  - $Y =$ all polys p with $p(\beta) = 0$

- φ not an auto (e.g. $\varphi(z) = \frac{1+z}{2}$):

  Reduce to automorphic case !!
  - φ an auto on
    "disc" $\Delta \supset \mathbb{U}$
  - C_φ HC on
    $$\underbrace{H^2(\Delta)}_{\text{dense, stronger norm}} \subset H^2(\mathbb{U})$$
  - C_φ HC on $H^2(\mathbb{U})$ $\square$

φ parabolic non-auto $\Rightarrow C_\varphi$ *not* HC.

Example. $\varphi(z) = \dfrac{1}{2-z}$

Proof (sketch). Suppose $\varphi(1) = 1$.

$$|1 - \varphi_n(z)| \sim 1 - |\varphi_n(z)| \sim \frac{\text{const.}}{n}$$

$\Rightarrow$ (after some work)

$$f(\varphi_n(z)) - f(\varphi_n(0)) \to 0 \quad \forall\ z \in \mathbb{U})$$

$\Rightarrow$

$$g(z) \equiv \text{const.}\ \forall\ g \in \overline{\text{orb}\,_{C_\varphi}(f)}$$ $\square$

$\varphi \notin$ LFT $(\mathbb{U})$

Linear Fractional Models

- (1880's) Koenigs (Fixed point in $\mathbb{U}$)
- (1930's) Valiron
  (1980's) Baker-Pommerenke, Cowen
  ("fixed point" on $\partial\mathbb{U}$)

Theorem (Bourdon-JHS 1993)

Under "regularity assumptions" on φ:

$$C_\varphi \text{ HC on } H^2 \iff C_\psi \text{ HC on } H^2$$

Linear Fractional model for φ

$$
\begin{array}{ccc}
\mathbb{U} & \xrightarrow{\varphi} & \mathbb{U} \\
\sigma \Big| & & \Big| \sigma \\
G & \xrightarrow{\psi} & G \\
\bigcap \Big| & & \Big| \bigcap \\
\mathbb{U} & \xrightarrow{\psi} & \mathbb{U}
\end{array}
\qquad
\begin{array}{ccc}
H^2(\mathbb{U}) & \xrightarrow{C_\varphi} & H^2(\mathbb{U}) \\
C_\sigma \Big| & & \Big| C_\sigma \\
H^2(G) & \xrightarrow{C_\psi} & H^2(G) \\
\bigcup \Big| & & \Big| \bigcup \\
H^2(\mathbb{U}) & \xrightarrow[C_\psi]{} & H^2(\mathbb{U})
\end{array}
$$

"Transference Thm."

$H^2(\mathbb{U})$ dense in $H^2(G)$

$\Rightarrow \quad C_\varphi$ HC on $H^2(\mathbb{U})$

C_ψ HC on $H^2(\mathbb{U})$

Walsh's Theorem (1930's):

∂G a Jordan curve

$\Rightarrow$ polynomials dense in $H^2(G)$

$\Rightarrow$ $H^2(\mathbb{U})$ dense in $H^2(G)$.

Defn. $\varphi : \mathbb{U} \to \mathbb{U}$ is *regular* if:

- φ unival. & contin. on $\overline{\mathbb{U}}$
- $\varphi(\overline{\mathbb{U}}) \subset \mathbb{U} \cup \{p\}$, where
- $p =$ Denjoy-Wolff point of φ

Cyclic behavior of C_φ: Denjoy-Wolff point at 1, $\varphi \in C^4(1)$, regular, and $\varphi''(1) \neq 0$.

Hypothesis on $\varphi'(p)$	Hypothesis on $\varphi''(p)$	Cyclicity of C_φ	Type of φ (Model for φ)
< 1	None	Hypercyclic	Hyperbolic
$=1$	Pure imag. $\neq 0$	Hypercyclic	Parabolic automorphism
$=1$	Real part > 0	Cyclic, Not Hypercyclic	Parabolic non-automorphis