II. Composition operator adjoints

1. *Last time*:

- $\mathbb{U} = \{|z| < 1\}$
- $\varphi : \mathbb{U} \to \mathbb{U}$ holomorphic.
- $C_\varphi : \mathrm{Hol}\,(\mathbb{U}) \to \mathrm{Hol}\,(\mathbb{U})$ defined by:

$$C_\varphi f = f \circ \varphi.$$

(a) *Littlewood (1925)*:

$$C_\varphi : H^2 \to H^2 \quad \text{bounded lin. op.}$$

(b) *Intertwining*: $XA = BX$ $(X \neq 0)$

- For $f, g \in H^\infty$:

$$C_\varphi T_g = T_h C_\varphi \iff g = h \circ \varphi.$$

- $(C_{z^2})^* T_{z^2} = T_z (C_{z^2})^*$,

  HOWEVER

- No *comp. op.* intertwines T_{z^2} and T_z.

2. *Question for today:* $\quad C_\varphi{}^* = ??$

(a) *Reproducing kernels.* For $a \in \mathbb{U}$:

$$K_a(z) \;:=\; \frac{1}{1 - a^* z} \;=\; \sum_{n=0}^{\infty} (a^*)^n z^n$$

Notation: $a^* \;:=\; \overline{a}$

(b) *Recall:* $\forall\ f \in H^2$:

$$f(a) = \langle f,\, K_a \rangle \qquad (a \in \mathbb{U})$$

(c) *Consequence:* $C_\varphi{}^* K_a = K_{\varphi(a)}$

Proof. $\forall\ a \in \mathbb{U},\ f \in H^2$:

$$\begin{aligned}
\langle f,\, C_\varphi{}^* K_a \rangle &= \langle C_\varphi f,\, K_a \rangle \\
&= \langle f \circ \varphi,\, K_a \rangle \\
&= f(\varphi(a)) \\
&= \langle f,\, K_{\varphi(a)} \rangle
\end{aligned}$$

□

(d) *Recall:* $\qquad T_f{}^* K_a = f(a)^* K_a$

3. *Cowen's Adjoint Thm.*

If $\varphi : \mathbb{U} \to \mathbb{U}$ *is linear fractional:*

$$\varphi(z) = \frac{az+b}{cz+d} \quad (\neq \text{ constant})$$

then $$C_\varphi{}^* = T_g \, C_\sigma \, T_h{}^*$$

where

- $h(z) = cz + d \quad \in H^\infty$
- $g(z) = \dfrac{1}{d^* - b^* z} \quad \in H^\infty \qquad (|d/b| = 1/|\varphi(0)| > 1)$
- $\sigma(z) = \dfrac{a^* z - c^*}{-b^* z + d^*} \quad : \mathbb{U} \to \mathbb{U}$

$\sigma = (\rho \circ \varphi^{-1} \circ \rho)$ *where*

$\rho(z) = 1/z^*$*; reflection in unit circle.)*

4. *Proof of Cowen's Theorem*

Given: $$\varphi(z) = \frac{az+b}{cz+d} : \quad \mathbb{U} \to \mathbb{U}$$

To Show: $$C_\varphi{}^* = T_g \, C_\sigma \, T_h{}^*$$

Enough to check this on *reproducing kernels* !

$$\begin{aligned}
C_\varphi{}^* K_z(w) &= K_{\varphi(z)}(w) = \frac{1}{1-\varphi(z)^* w} \\
&= \frac{1}{1-\left(\frac{a^*z^*+b^*}{c^*z^*+d^*}\right) w} \\
&= \frac{c^*z^*+d^*}{(c^*z^*+d^*)-(a^*z^*+b^*)w} \\
&= \frac{c^*z^*+d^*}{(c^*-a^*w)z^*+(d^*-b^*w)} \\
&= \frac{1}{d^*-b^*w} \quad \frac{1}{1-\left(\frac{a^*w-c^*}{-b^*w+d^*}\right) z^*} \quad (cz+d)^* \\
&= \quad g(w) \qquad K_z(\sigma(w)) \qquad h(z)^* \\
&= T_g \; C_\sigma \; T_h{}^* \; K_z
\end{aligned}$$

$\square$

5. *Beyond Linear-fractional – I*

Simplest case: $\varphi(z) = z^2$

$\forall\ f, g \in H^2$:

$$\langle (C_{z^2})^* f,\, g\rangle \;=\; \langle f,\, C_{z^2}\, g\rangle$$

$$= \;\langle \sum \hat{f}(n)\, z^n, \sum \hat{g}(n)\, z^{2n}\rangle$$

$$= \;\sum \hat{f}(2n)\, \hat{g}(n)^*$$

$$= \;\langle \sum \hat{f}(2n)\, z^n, \sum \hat{g}(n)\, z^n\rangle$$

Therefore $\quad \forall\ f \in H^2$:

$$(C_{z^2})^* f(z) \;=\; \sum \hat{f}(2n)\, z^n$$

$$= \;\frac{1}{2}[f(\sqrt{z}) + f(-\sqrt{z})]$$

i.e.,

$$C_{z^2} = \text{“}\, \frac{1}{2}(C_{\sqrt{z}} \;+\; C_{-\sqrt{z}})\, \text{''}$$

6. *Beyond Linear-fractional – II*

$\varphi : \mathbb{U} \to \mathbb{U}$ *rational*.

- McDonald 2003 (Blaschke products)
- Martín–Vukotić 2006 (Residues)
- Cowen–Gallardo 2006
  ("mult-valued" wtd. comp. ops.)
- Hammond–Moorhouse–Robbins (HMR) 2007
  (complete solution)
- Bourdon–Shapiro 2008 "Branch-less" proof of (HMR)

TODAY: *The HMR formula.*

Thm (HMR). *If $\varphi : \mathbb{U} \to \mathbb{U}$ rat'l, degree $d \geq 1$:*

$$C_\varphi{}^* f(z) \;=\; \frac{f(0)}{1 - \varphi(\infty)^* z} \;+\; \sum_{j=1}^{d} g_j(z) f(\sigma_j(z))$$

where

- $z \in \mathbb{U} \setminus \{$*finite set*$\}$,
- $\{\sigma_1(z), \ldots, \sigma_d(z)\} = (\rho \circ \varphi \circ \rho)^{-1}(\{z\})$,
- $g_j(z) = z\,\frac{\sigma_j'(z)}{\sigma_j(z)}$.

7. <u>*HMR continued*</u>

Thm. $\varphi : \mathbb{U} \to \mathbb{U}$ rat'l, degree $= d$.

- $\varphi_e = \rho \circ \varphi \circ \rho : \mathbb{U}_e \to \mathbb{U}_e$.
- $z \in \mathbb{U} \setminus \{\varphi_e(0)\}$ *regular* for φ_e.
- $\varphi_e^{-1}(\{z\}) \;=\; \{\sigma_1(z), \ldots, \sigma_d(z)\}$.
- $g_j(z) = z\,\dfrac{\sigma_j'(z)}{\sigma_j(z)}$.

$$\Rightarrow$$

$$C_\varphi{}^* f(z) \;=\; \frac{f(0)}{1 - \varphi(\infty)^* z} \;+\; \sum_{j=1}^{d} g_j(z) f(\sigma_j(z))$$

Example. $\varphi(z) \equiv z^2$

- $\varphi_e(z) \equiv z^2$, so $\varphi_e(\infty) = \infty$ and
- $\sigma_1(z) = \sqrt{z}$, $\sigma_2(z) = -\sqrt{z}$
- $g_j(z) \equiv 1/2$.

$$\Rightarrow$$

$$C^*_{z^2} f(z) \;=\; \tfrac{1}{2} f(\sqrt{z}) \;+\; \tfrac{1}{2} f(-\sqrt{z})$$

8. *HMR continued*

Thm. $\varphi : \mathbb{U} \to \mathbb{U}$ rat'l, degree $= d$.

- $\varphi_e = \rho \circ \varphi \circ \rho : \mathbb{U}_e \to \mathbb{U}_e$.
- $z \in \mathbb{U} \setminus \{\rho(\varphi(\infty))\}$ *regular* for φ_e.
- $\varphi_e^{-1}(\{z\}) \ = \ \{\sigma_1(z), \ldots, \sigma_d(z)\}$.
- $g_j(z) = z \, \dfrac{\sigma_j'(z)}{\sigma_j(z)}$.

$$\Longrightarrow$$

$$C_\varphi{}^* f(z) \ = \ \frac{f(0)}{1 - \varphi(\infty)^* z} \ + \ \sum_{j=1}^{d} g_j(z) f(\sigma_j(z))$$

"Wishful Thinking Formula"

$$\text{(WTF)} \qquad C_\varphi{}^* \ = \ \Lambda_\infty \ + \ \sum_{j=1}^{d} T_{g_j} C_{\sigma_j}$$

Qn. When is (WTF) "legitimate?"

NOT ALWAYS: e.g., $\varphi(z) = z^2$!!

9. *Example:* (WTF) *"legitimate"*

$$\varphi(z) = \frac{1}{3 - z - z^2}$$

$$\varphi_e(w) \ = \ 1/\varphi(1/w^*)^* \ = \ 3 - \frac{1}{w} - \frac{1}{w^2}$$

Solve $z = \varphi_e(w)$ to get $w = \sigma_j(z)$

$$\sigma_1(z) \quad = \quad \frac{1 + \sqrt{13 - 4z}}{2(2 - z)}$$

$$\sigma_2(z) \quad = \quad \frac{1 - \sqrt{13 - 4z}}{2(2 - z)}$$

where "$\sqrt{\ }$" = principal branch.

- σ_1, σ_2 holomorphic on $\mathbb{C} \setminus (\frac{13}{4}, \infty)$
- $\varphi(\infty) = 0$, so $\Lambda_\infty : f \to f(0)$

$$C_\varphi{}^* \ = \ \Lambda_\infty \ + \ T_{g_1} C_{\sigma_1} \ + \ T_{g_2} C_{\sigma_2}$$

a "legitimate operator equation."

The Key: Critical *values* of φ all lie in $\mathbb{U}$!!

10. <u>*"Branch-less" proof of HMR*</u>

$$C_\varphi{}^* f(z) \;=\; \frac{f(0)}{1-\varphi(\infty)^* z} \;+\; \sum_{j=1}^{d} g_j(z) f(\sigma_j(z))$$

(a) $C_\varphi{}^* f(z) = \langle C_\varphi{}^* f,\, K_z\rangle = \langle f,\, C_\varphi K_z\rangle = \langle f,\, K_z \circ \varphi\rangle$

(b) $C_\varphi{}^* f(0) = f(0)$

(c) Fix $z \in \mathbb{U} \setminus \{0, \rho(\varphi(\infty))\}$, reg. value of φ_e:

(d) $K_z(\varphi(w)) = \dfrac{1}{1 - z^*\varphi(w)}$

distinct poles $\{w_1, w_2, \ldots, w_d\} = \varphi^{-1}(\{\rho(z)\})$

$$\begin{aligned}
K_z \circ \varphi(w) &= \alpha + \sum_{j=1}^{d} \frac{\beta_j}{w - w_j} \qquad (K_z(\varphi(\infty)) \neq \infty,\; w_j \in \mathbb{U}_e) \\
&= \alpha - \sum_{j=1}^{d} \frac{\beta_j / w_j}{1 - w/w_j} \\
&= \alpha - \sum_{j=1}^{d} \frac{\beta_j\, \rho(w_j)^*}{1 - \rho(w_j)^* w} \qquad (\rho(w_j) \in \mathbb{U}) \\
&= \alpha - \sum_{j=1}^{d} \beta_j\, \rho(w_j)^* K_{\rho(w_j)}(w)
\end{aligned}$$

$$C_\varphi{}^* f(z) \;=\; \frac{f(0)}{1-\varphi(\infty) z^*} \;-\; \sum_{j=1}^{d} \beta_j{}^* \rho(w_j) f(\rho(w_j))$$

11. *Computation of β_j's*

Goal: $C_\varphi{}^* f(z) \;=\; \dfrac{f(0)}{1-\varphi(\infty)^* z} \;+\; \displaystyle\sum_{j=1}^{d} g_j(z) f(\sigma_j(z))$

So far:

- $C_\varphi{}^* f(z) \;=\; \langle f,\, K_z \circ \varphi \rangle$
- $K_z(\varphi(w)) \;=\; \alpha + \displaystyle\sum_{j=1}^{d} \frac{\beta_j}{w - w_j}$ (finite at ∞, simple poles)
- $C_\varphi{}^* f(z) \;=\; \dfrac{f(0)}{1-\varphi(\infty) z^*} \;-\; \displaystyle\sum_{j=1}^{d} \beta_j{}^* \rho(w_j) f(\rho(w_j))$

(b) Compute β_j's—residues!

$$
\begin{aligned}
\beta_j &= \lim_{w \to w_j} K_z(\varphi(w)) \\
&= \lim_{w \to w_j} \frac{w - w_j}{[1 - z^*\varphi(w)] - [1 - z^*\varphi(w_j)]} \\
&= \left(\frac{d}{dw} [1 - z^*\varphi(w)]_{w = w_j} \right)^{-1} \\
&= -(z^*\varphi'(w_j))^{-1} \qquad \square
\end{aligned}
$$

So far:

$$C_\varphi{}^* f(z) = \frac{f(0)}{1-\varphi(\infty)^* z} + \sum_{j=1}^{d} \frac{\rho(w_j)}{z\,\varphi'(w_j)^*} f(\rho(w_j))$$

12. *Endgame*

Goal: $C_\varphi{}^* f(z) = \dfrac{f(0)}{1-\varphi(\infty)^* z} + \sum_{j=1}^{d} g_j(z)\, f(\sigma_j(z))$

So far:

$$C_\varphi{}^* f(z) \;=\; \frac{f(0)}{1-\varphi(\infty)^* z} \;+\; \sum_{j=1}^{d} \frac{\rho(w_j)}{z\,\varphi'(w_j)^*}\, f(\rho(w_j))$$

- $\{w_j\}_1^d = \{\text{poles of } K_z(\varphi(w))\} = \varphi^{-1}(\{\rho(z)\})$
- $\{\rho(w_j)\} = \varphi_e^{-1}(\{z\}) = \{\sigma_1(z), \ldots \sigma_d(z)\}$

$$\Longrightarrow$$

$$C_\varphi{}^* f(z) = \frac{f(0)}{1-\varphi(\infty)^* z} + \sum_{j=1}^{d} \frac{\sigma_j(z)}{z\,\varphi'(\rho(\sigma_j(z)))^*}\, f(\sigma_j(z))$$

<u>To Show</u>:

$$g_j(z) \;:=\; \frac{z\sigma_j'(z)}{\sigma_j(z)} \;=\; \frac{\sigma_j(z)}{z\,\varphi'(\rho(\sigma_j(z)))^*}$$

13. *End of endgame*

Given: $\varphi_e(\sigma(z)) \equiv z, \qquad \varphi_e = \rho \circ \varphi \circ \rho$

Show:
$$\frac{\sigma(z)}{z\,\varphi'(\rho(\sigma(z))^*} \;=\; z\,\frac{\sigma'(z)}{\sigma(z)}$$

Proof. Write $\varphi^*(z) = \varphi(z^*)^*$.

$$\begin{aligned}
\varphi_e(\sigma(z)) \equiv z \quad &\Longleftrightarrow \quad \varphi^*(1/\sigma(z)) = 1/z \\
&\Rightarrow \quad 1/z^2 = \varphi^{*\prime}(1/\sigma(z))\,\sigma(z)^{-2}\,\sigma'(z) \\
&\qquad\qquad = \varphi'(1/\sigma(z)^*)^*\,\sigma(z)^{-2}\,\sigma'(z) \\
&\qquad\qquad = \varphi'(\rho(\sigma(z))^*\,\sigma(z)^{-2}\,\sigma'(z)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \frac{\sigma(z)}{z\,\varphi'(\rho(\sigma(z))^*} \;&=\; \frac{\sigma(z)}{z}\,\frac{z^2\sigma'(z)}{\sigma(z)^2} \\
&=\; z\,\frac{\sigma'(z)}{\sigma(z)}
\end{aligned}$$

$\square$

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